Aspects of plane waves and Taub-NUT as exact string theory solutions

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Abstract

This thesis is a study of some aspects of string theory solutions that are exact in the inverse string tension α', and thus are valid beyond the low-energy limit.

I investigate D-brane interactions in the maximally supersymmetric plane wave solution of type IIb string theory, and study the fate of the stringy halo surrounding D-branes. I find that the halo is like in flat space for Lorentzian D-branes, while it receives a non-trivial modification for Euclidean D-branes.

I also comment on the connection between Hagedorn temperature and T-duality, which motivates a more general study of T-duality in null directions. I consider such transformations in a spinning D-brane solution of supergravity, and find that divergences in the field components associated with null T-dualities are invisible to string and brane probes. I also observe that there are closed timelike curves in all the T-dual solutions, but that none of them are geodesics.

The second half of the thesis is an investigation of the fate of closed timelike curves and of cosmological singularities in an exact stringy Taub-NUT solution of heterotic string theory, and in a rotating generalisation of it. I compute the exact spacetime fields, using a description in terms of a gauged Wess-Zumino-Novikov-Witten model, and find that the α' corrections are mild. The key features of the Taub-NUT geometry persist, together with the emergence of a new region of space with Euclidean signature. Closed timelike curves are still present, which is interpreted as a sign that they might be a natural ingredient in string theory, for instance in pre-Big-Bang cosmological scenarios.

Declaration

The work in this thesis is based on research carried out at the Centre for Particle Theory, the Department of Mathematical Sciences, University of Durham, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

Parts of the work presented in this thesis have been published as research papers written in collaboration with my supervisor, Prof. Clifford V. Johnson. These are refs. [1,2], and correspond to chapters 3 and 6. Chapters 2 and 5 are mostly reviews of already known material, while chapters 4 and 7 contain new unpublished results based on my own research.

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The work towards the completion of this thesis started in many ways long before I came to Durham. And ever since my first day in primary school, I have had excellent teachers who have inspired me and encouraged me. A great part of the achievement manifested by this thesis belongs to you.

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Chapter 1

Introduction

1.1 State of string theory

String theory appeared first [3–6] as a theory to explain strong nuclear interactions. It was put aside as such by the successful Quantum Chromo Dynamics (QCD) theory. However, it became clear that string theory is a theory of a much greater potential [7]: a theory of quantum gravity. The theory as far as it was understood had various problems, and much of the initial excitement went away. But a decade later a new breakthrough was made with the discovery of anomaly cancellations [8], often referred to as the *first string revolution.* Since then, much work has been done in string theory, with a number of remarkable advances. One of the most important came in the mid-nineties with the discovery of D-branes [9], setting off the *second string revolution.*

Insight gained over the last decade or so has taught us that the different superstring theories (type I, type IIA and B, heterotic $E_8 \times E_8$ and $SO(32)$) are really just different perturbative limits of the same underlying theory, the *M-theory.* But as clear as this is, it is equally unclear what is the proper formulation of this mysterious theory.

A summary of my thesis for laymen is given in appendix B. An excellent introduction to string theory for non-physicists is ref. [10]. Textbooks on string theory are refs. [11–15]. Some other, briefer introductions are refs. [16–19].

String theory and the real world

In many respects it might be more fruitful to think of string theory as a framework for theoretical physics, rather than as one particular theory about the fundamental particles and their interactions. String theory has and continues to teach us a lot about connections within theoretical physics and indeed mathematics. For example, string theory has through various dualities opened up windows to exact non-perturbative phenomena in gauge theory. This makes string theory highly interesting to study, independently of whether it is an accurate description of elementary particles and gravity as manifested in our world. So there are several strong, but purely theoretical motivations for doing string theory. From this point of view, the research is not so much about understanding the universe per se, but about understanding our theories about the universe. By understanding them better, we can more easily extend them, or correct them where they turn out faulty or inconsistent.

Nevertheless, much work is also being undertaken to try and relate string theory to the observable world. Phenomenology is the bridging discipline between theoretical and experimental physics, and over the last few years the new field of string phenomenology has evolved. The central questions are: What can we deduce from string theory about the observable world? What constraints do our experimental results put on string theory? Of utmost interest is the question of how to reproduce the Standard Model from string theory. In the "old days" (before mid-nineties), heterotic strings were the favourite tool to achieve low-energy dynamics in four dimensions that resemble our world. Nowadays, type II theories with D-branes are the more popular ones – but with the understanding that all the superstring theories are anyway related by various dualities.

Another area where there are real prospects for communication between string theory and observations is cosmology. Recent years have seen remarkable progress in astronomy, with increased accuracy in observational data. These put strong constraints on various fundamental parameters. They also seem to tell us that the vast majority of the energy in the universe is of a still unknown sort called "dark energy". String theory should predict these data if it is to be taken seriously as a quantum gravity theory. Moreover, the energy scales needed to test string theory seem to be far too high to be achievable in a laboratory on Earth, so data taken from (indirect) observations of extreme phenomena like the first moments of the universe, and of black holes might be our best chance as a test ground for string theory.

Having said this, certain aspects of string theory should also be testable in the next generation accelerators.[1] The most important amongst these is probably supersymmetry, which can be viewed as a prediction from string theory. If supersymmetry is not detected, it will be an embarrassment, but certainly not the end of string theory.

[1]The Large Hadron Collider at CERN is to be operational in 2007.

Some open problems

One of the big problems in string theory is the vacuum degeneracy: There is an enormously large number of different ways to arrive at low-energy dynamics in four dimensions, and no good principle to tell us which is the right one. This is related to the cosmological constant problem: Why is it that the cosmological constant is so small, yet non-zero and positive? The vacuum degeneracy is also at the heart of the problems with making experimental predictions from string theory. The problem is not that string theory gives wrong predictions, but rather that it does not single out any one of the large number of possibilities.

On a conceptual level, a problem with string theory today is that it does not really have a proper definition. The string theories are only defined perturbatively, and although there is strong evidence of the existence of an underlying theory, the M-theory, its proper formulation is still unknown. These and other difficulties usually makes it hard to do calculations in the full string theory. One way to get a much more tractable theory is to restrict ourselves to the low-energy regime where the inverse string tension $\alpha' \to 0$. Most explicit calculations in string theory have been done in this limit. This thesis is a study of some aspects of string theory *beyond* this low-energy limit.

There are of course many more open questions and unsolved problems in string theory, but it is a much too involved task to discuss them all here.

1.2 Context and motivation

Much is known about string theory and solutions of the string theory equations in the low-energy $\alpha' \to 0$ regime, which is the supergravity limit. The title of this thesis includes the words "exact string theory solutions", and by that I mean solutions that go beyond this limit, and are valid solutions of string theory to all orders in α', including non-perturbative effects. Note that the supergravity limit sends the string length $l_s = \sqrt{\alpha'}$ to zero, so effectively it treats the strings as point particles and ignores the oscillations of the strings. This gives a field theory of particles, but it nevertheless contains stringy corrections through the presence of extra fields, like the antisymmetric B-field and the Ramond-Ramond fields.

A common feature of supergravity solutions, which I will discuss in this thesis, is the existence of closed timelike curves (CTCs). These are also there in General Relativity, and represent time-loops in the spacetimes, leading to problematic paradoxes and divergences in physical quantities like the stress-energy tensor. It is a common belief that the CTCs and the problems associated with them disappear in the full string theory. But due to the lack of string models where CTCs can be studied by explicit calculations, this has mainly remained a speculation.

It is clear that the string oscillations cannot always be ignored, and it is interesting to study string theory models where we can also take into account that the strings are not pointlike objects. However, the scarcity of exact classical solutions where this is possible is an obstacle to a better understanding of string theory. There are only two classes of solutions known to be exact in α'. The first is the class of plane-wave backgrounds, or more generally, solutions with a covariantly constant null Killing vector. These are exact solutions which receive no α' corrections due to special properties of the background.

The second is the class of gauged Wess-Zumino-Novikov-Witten (WZNW) models. These are worldsheet conformal field theories which have a Lagrangian description which makes it relatively easy to deduce the quantum effective action, and thereby the exact background fields.

In addition, Minkowski space is an exact string theory solution, and also orbifolds of Minkowski space are. These are exact because the curvature is zero, and therefore all α' corrections vanish to all orders. An example [20] of such an orbifold is the Milne space. For a review of exact solutions of closed string theory see ref. [21].

In this thesis I will study examples from each of the two primary classes of exact solutions. The first is a parallel plane wave with non-zero Ramond-Ramond (RR) flux. The aspect of it that I am going to investigate concerns D-brane–anti-D-brane interactions, in particular the so-called stringy halo surrounding the D-branes.

The gauged WZNW model examples I will study are a stringy Taub-NUT space and a generalisation of it with rotation. I will compute the α' corrections to the spacetime fields (which have not been computed before), and then study some properties of the solution.

1.3 Outline and summary of results

In chapters 2 and 3 I will study D-branes and D-brane interactions in a plane wave background. D-branes are at present well understood in a flat spacetime background, but rather poorly understood in more general backgrounds. Being a background where the study of D-branes *is* tractable, the plane waves are therefore of great interest in this respect. D-branes are known to be surrounded by a stringy halo, which marks the edge of the region within which tachyon condensation occurs. The goal of the first part of my thesis is to investigate this halo in the plane wave background, and study how it differs from what is known for D-branes in flat spacetime.

We shall see that the there is an important difference between Lorentzian D-branes and Euclidean D-branes. The former have a stringy halo just like in the flat space, while the latter have a halo deformed by the plane wave parameter μ.

Chapter 4 picks up on an observation regarding thermal strings in plane waves: T-dualties in null or timelike directions might be relevant for understanding the Hagedorn temperature. In this chapter I will study another system to gain more insight into the general issue of T-dualities in null and timelike directions. An asset of the model I will consider is that it contains a parameter which serves as a regulator, allowing the T-duality to become null or timelike in a controlled way.

I find that probe calculations give reasons to believe that null or timelike T-dualities do not really represent a problem in the string theory, although components of the supergravity fields become divergent. We will also observe closed timelike curves in this model, persisting the T-dualities, and I argue that none of these are geodesics.

The observation of closed timelike curves in supergravity solutions, and the wish to understand their role in string theory is part of the motivation for the work of the subsequent chapters 5, 6 and 7. I will do this by studying two examples from the second class of exact string theory solutions. These are gauged Wess-Zumino-Novikov-Witten models, associated with heterotic coset model constructions. I will present the general techniques required to deduce the exact spacetime fields (chapter 5), and then study a stringy Taub-NUT solution (chapter 6) and its rotating generalisation (chapter 7).

Of special interest is the issue of CTCs in the resulting spacetimes. The corresponding supergravity solutions have been known to have CTCs, and these models provide exactly the kind of computational control we want for an investigation of the fate of CTCs in full string theory. Another motivation for investigating these models is that they contain cosmological regions with a Big Bang and a Big Crunch, thereby facilitating a study of cosmological singularities in string theory.

The conclusion is that the string theory modifications are mild and do not modify the spacetime in such a way that the closed timelike curves disappear. This is a hint that although problematic in General Relativity, the existence of closed timelike curves might not necessarily be a problem in string theory.

Chapter 2

Plane wave solutions

Plane wave spacetimes are exact solutions of superstring theory which may or may not come equipped with Neveu-Schwarz-Neveu-Schwarz (NSNS) or Ramond-Ramond (RR) flux. In this and the next chapter, I will focus on plane waves with a non-trivial five-form RR field [22]. They are very interesting for various reasons. First of all, they are tractable systems which are exact to all orders in α', and therefore very useful models in which to study string theory in RR backgrounds, which has been a poorly understood subject. Moreover, the parallel plane wave (pp-wave), which has *constant* RR flux, contains 32 supersymmetries, which is the maximum number. This puts it alongside Minkowski space and the $AdS_5 \times S^5$ solutions as the only maximally supersymmetric solutions of string theory. In fact, the pp-wave can be found as a Penrose-Güven limit of $AdS_5 \times S^5$, and this is another highly attractive asset of it, as I will come back to. It means that the plane wave plays part in string/gauge theory dualities inherited from the AdS/CFT correspondence via the Penrose-Güven limit.

To find the spectrum of states we have to rely on light-cone gauge, which then makes the task easy. But that gauge also has some awkward features. It means that the worldsheet theory is not conformally invariant since the worldsheet fields are massive. So *e.g.*, string interactions are hard to evaluate. However, in this and the next chapter I will study static interactions between pairs of D-branes, which is an easier problem.

I will start with a review of string theory in the plane wave background. Then I will briefly describe the AdS/CFT correspondence, and the limit of it that is relevant for plane wave physics. The main part of the chapter is a general discussion of D-brane interactions and the open/closed duality. This will set up the notation, and serve as background for the discussion in chapter 3. Finally, I will demonstrate the existence of a Hagedorn temperature and its relation to the self-dual radius under a T-duality transformation.

2.1 Strings in plane waves

The parallel plane wave spacetime is a maximally supersymmetric solution of string theory with the metric and RR flux given as [22]:

$$
\begin{aligned}
ds^2 &= 2dx^+dx^- - \mu^2 x^2 (dx^+)^2 + \sum_{i=1}^{4} dx^i dx^i + \sum_{i=5}^{8} dx^i dx^i \ , \\
F_{+1234} &= F_{+5678} = 2\mu \ , \quad x^2 = \sum_{i=1}^{8} x^i x^i \ , \quad x^\pm = \frac{1}{\sqrt{2}}(x^9 \pm x^0) \ .
\end{aligned}
\tag{2.1}
$$

It yields an exactly solvable string model [23] (in light-cone gauge). The metric has an $SO(8)$ symmetry which is broken to $SO(4) \times SO(4) \times \mathbb{Z}_2$ by the presence of the RR-field.

2.1.1 Quantisation

The presence of the RR flux in the plane wave solution (2.1) makes it highly non-trivial to formulate string theory in this background. But if we ignored this problem and tried to write down a standard nonlinear sigma model,

$$
S \sim \frac{1}{4\pi\alpha'} \int d^2\sigma \Big[\sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) + (\text{fermions}) + (\text{RR fields}) \Big], \tag{2.2}
$$

even the bosonic sector would be problematic in a general covariant gauge, due to the non-trivial spacetime metric $G_{\mu\nu}$. However, if we choose a *light-cone gauge* defined by relating worldsheet time τ to x^+ via $x^+ = 2\pi\alpha' p^+ \tau$, where p^+ is the $+$ component of spacetime momentum, the bosonic part simplifies and essentially gives a free massive theory.

Remarkably, the light-cone gauge also allows us to write down the contribution from the fermionic sector [23]. This is done using the Green-Schwarz formulation of superstrings, rather than the more conventional Ramond-Neveu-Schwarz formulation. The resulting Lagrangian is

$$
\mathcal{L} = \frac{1}{4\pi\alpha'}(\partial_+ x^i \partial_- x^i - M^2 x^2) + \frac{i}{2\pi\alpha'}(S^a \partial_+ S^a + \tilde{S}^a \partial_- \tilde{S}^a - 2M S^a \Pi_{ab} \tilde{S}^a) \ , \tag{2.3}
$$

where $i = 1, \ldots, 8$ is a spacetime index, $a = 1, \ldots, 8$ is a spinor index, and the mass parameter is $M = 2\pi\alpha' p^+ \mu$. The matrix Π is a product of gamma matrices,

$\Pi = \gamma^1\gamma^2\gamma^3\gamma^4$, and the spinors S^a and $\tilde{S}^a$ are left- and right-moving $SO(8)$ *spacetime* spinors respectively.

It is apparent from the Lagrangian (2.3) that this is a model of eight free massive bosons and fermions, and should therefore be easy to quantise. The light-cone (closed string) Hamiltonian is

$$2p^+H = \sum_{k=-\infty}^{\infty} |\omega_k| N_k, \qquad \omega_k = \mathrm{sign}(k)\sqrt{k^2 + M^2}, \tag{2.4}$$

where N_k is the total (bosonic plus fermionic) number operator, and ω_k is the oscillator frequency for mode k. A notable difference from flat space is that also the zero-modes are oscillators in the plane wave case. Upon quantisation, the Hamiltonian (2.4) gives the complete spectrum for the strings in the plane wave background, and the model is therefore said to be an *exactly solvable* string theory model.

2.1.2 Plane waves as a Penrose-Güven limit

An interesting feature of the plane wave solution (2.1) is that it can be found as a particular scaling limit of $AdS_5 \times S^5$ [24]. This is the Penrose-Güven limit [25, 26], which essentially blows up the geometry of any spacetime around a chosen null geodesic, while scaling the various other fields such that they also survive the limiting procedure. Such limits always result in plane wave spacetimes, albeit more general ones than the *parallel* plane wave (pp-wave) solution (2.1) that I consider in this thesis.

How to take this limit can be formulated in general terms, but in the case of $AdS_5 \times S^5$ a rather direct and simple procedure is possible.[1] The $AdS_5 \times S^5$ metric can be written

$$ds^2 = R^2\Big[-\cosh^2\!\rho\; dt^2 + d\rho^2 + \sinh^2\!\rho\; d\Omega_3^2 + \cos^2\!\theta\; d\psi^2 + d\theta^2 + \sin^2\!\theta\; d\Omega_3'^2\Big], \tag{2.5}$$

where R is the radius of curvature, and is the same for both parts of the metric. Introduce light-cone coordinates $\tilde{x}^\pm = \frac{1}{\sqrt{2}}(\psi \pm t)$. Then make the rescalings

$$\tilde{x}^+ = \mu x^+, \quad \tilde{x}^- = \frac{x^-}{\mu R}, \quad \rho = \frac{r}{R}, \quad \theta = \frac{y}{R}, \tag{2.6}$$

with an arbitrary mass parameter μ, and take the limit $R \to \infty$. This "blows up"

[1] A calculation demonstrating how the limit can be taken in the more general case is found in ref. [27].

the region close to the null-geodesic parameterised by $\tilde{x}^+$. The resulting metric is

$$ds^2 = 2dx^+dx^- - \mu^2(r^2+y^2)(dx^+)^2 + (dr^2 + r^2 d\Omega_3^2) + (dy^2 + y^2 d\Omega_3'^2). \tag{2.7}$$

The $AdS_5 \times S^5$ spacetime as a solution of type IIB string theory also comes with a RR flux

$$F_5 = \epsilon(AdS_5) + \epsilon(S^5), \tag{2.8}$$

where $\epsilon(M)$ is the natural volume form on the manifold M. In the above scaling limit this becomes

$$F_5 = 2\mu\big(dx^+ \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 + dx^+ \wedge dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8\big). \tag{2.9}$$

Hence the Penrose-Güven limit of the $AdS_5 \times S^5$ solution of type IIB string theory gives exactly the plane wave solution (2.1).

With the AdS/CFT correspondence in mind (see next section), this is very interesting. We know that there is a duality between string theory on $AdS_5 \times S^5$ and a certain conformally invariant gauge theory. However, one problem with the AdS/CFT correspondence is that the spectrum of states is unknown for string theory in $AdS_5 \times S^5$, so that for practical computations we are restricted to the supergravity limit.

But we have just seen that the plane wave background is tractable even for the full string theory, and since it is related to $AdS_5 \times S^5$ by a scaling limit, there is good reason to be excited: There ought to be a *dual limit* on the gauge theory side, giving a correspondence between string theory on plane waves and some sector of a gauge theory. Such a correspondence indeed exists [28], and is often referred to as the BMN limit of the AdS/CFT correspondence. It opens up a whole new window to string/gauge theory dualities, and has been a field of active research since its discovery.

2.2 The AdS/CFT correspondence and its BMN limit

2.2.1 AdS/CFT correspondence

The AdS/CFT correspondence [29–31] is a realisation of quite an old idea [32] that there is a connection between large N gauge theories and string theories. It is also a realisation of the holographic principle that a gravitational theory in a number of dimensions is equivalent to a non-gravitational theory in less dimensions. In short, the AdS/CFT correspondence states:

> Type IIB string theory on $AdS_5 \times S^5$ with integer 5-form flux N across the S^5 is *dual* to $\mathcal{N} = 4$ supersymmetric conformal field theory (CFT) in 3+1 dimensions with gauge group $U(N)$.

Recall that type IIB string theory has D-branes of odd spatial dimensions. To derive a weaker form of the correspondence, consider an integer number N of coincident D3 branes in flat 10-dimensional Minkowski space. There are two different low-energy descriptions of this system, and it is the equivalence of these descriptions which gives the above correspondence.

First, however, let me clarify what limit we shall be working in: Remember that the low-energy approximation of string theory is obtained by sending $\alpha' \longrightarrow 0$. We want to arrive at a sensible gauge theory where certain quantities survive the limit, e.g. the energy of a "W boson". Now, consider a brane located away a distance r from the N coincident D3-branes. A string stretched between this brane and the brane-stack represents such a "W boson" with an energy = tension×length $\sim \frac{r}{\alpha'} \equiv u$. So, this is a quantity we want to keep fixed, which means we have to send $r \longrightarrow 0$.

Description one The system we consider can be described by open and closed strings. The D3-branes represent excitations of open strings, while the bulk geometry represents excitations of closed strings. In addition, there are interactions between the brane and the bulk. In the limit $\alpha \longrightarrow 0, r \longrightarrow 0, \frac{r}{\alpha'}$ = fixed, the open string excitations are described by a gauge theory living on the brane worldvolume. To be more specific, it is an $\mathcal{N} = 4$ supersymmetric $U(N)$ conformally invariant gauge theory (CFT) in 3+1 dimensions. The gauge coupling is $g^2_{YM} = 2\pi g_s$, where g_s is the string coupling.

Now, consider the closed string excitations in the bulk. By taking the low-energy limit $\alpha' \longrightarrow 0$, we are left with the massless modes only, that is, type IIB supergravity.

The interactions between these two sectors are suppressed in the limit. For this reason it is referred to as the *decoupling limit.* This decoupling of the open and closed sectors is crucial for proving the weak correspondence.

In summary, this limit means

$$S = S_{\text{brane}} + S_{\text{bulk}} + S_{\text{int}} \quad \longrightarrow \quad S_{\text{CFT}} + S_{\text{flat space sugra}}. \tag{2.10}$$

Description two Another way to view the system of N coincident D3-branes is by realising that it corresponds to a solitonic solution of type IIB supergravity. The branes warp the geometry, and are sources for a RR 5-form field strength F_5. The bosonic part of this solution is given as

$$
\begin{aligned}
ds^2 =& H(r)^{-\frac{1}{2}}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + H(r)^{\frac{1}{2}}(dr^2 + r^2 d\Omega_5^2), \\
F_5 =& (1+*)dC_4, \quad C_4 = dt \wedge dx_1 \wedge dx_2 \wedge dx_3, \\
e^{2\Phi} =& g_s^2,
\end{aligned}
\tag{2.11}
$$

where

$$
H(r) = 1 + \frac{R^4}{r^4}, \qquad R^2 \equiv \alpha' \sqrt{4\pi g_s N}. \tag{2.12}
$$

It is easy to see that the above solution has a horizon for $r = 0$, so the limit $r \longrightarrow 0$ means that we zoom in on the near horizon geometry. Doing this with $u = \frac{r}{\alpha'}$ fixed gives

$$
ds^2 = \frac{u^2}{R^2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + R^2(\frac{du^2}{u} + d\Omega_5^2), \tag{2.13}
$$

where a factor of α' has been absorbed into the coordinates t and x_i. This geometry is $AdS_5 \times S^5$ (in local coordinates), with the same radius of curvature R for both the anti-de Sitter part and of the 5-sphere.

Again there is a decoupling, now between this near-horizon region and the asymptotic region. As $r \to \infty$, the solution (2.11) gives just flat Minkowski space, and so this asymptotic region is described by IIB supergravity on flat space. There are no interactions left in this limit, since there would be an infinite redshift for a signal going from the near horizon region to the asymptotic region.

In summary, the limit in this description amounts to

$$
S = S_{\text{warped D3-brane geometry}} \quad \longrightarrow \quad S_{\text{AdS sugra}} + S_{\text{flat space sugra}}. \tag{2.14}
$$

Comparing the two descriptions, we see that in both cases we have a part described by supergravity in flat space. Identifying these, we are led also to identify the two other terms. Hence we identify 4D $U(N)$ gauge theory (which is a CFT) with type IIB supergravity in $AdS_5 \times S^5$ background.

This gauge theory is a rather special one in that it is has $\mathcal{N} = 4$ superconformal symmetry. The bosonic part of the Lagrangian is

$$
\mathcal{L} = \mathrm{Tr}\Big[-\frac{1}{2g_{YM}^2} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\sum_{i=1}^{6} D_\mu \phi_i D^\mu \phi^i + \frac{g_{YM}^2}{4}\sum_{ij}[\phi^i, \phi^j]^2\Big], \tag{2.15}
$$

where $F_{\mu\nu}$ is the gauge field strength, and ϕ^i are six real scalars. The fields belong to a supermultiplet which transforms in the adjoint of $U(N)$. The $\mathcal{N} = 4$ superalgebra is very constraining, and determines the Lagrangian almost uniquely. The only freedom is in the gauge group and the coupling g_{YM} which is a true (non-running) parameter since the β-function vanishes due to the conformal symmetry.

Validity of descriptions The supergravity approximation to string theory can only be trusted when curvatures are small (*i.e.*, radius of curvature is large compared to string length $\sqrt{\alpha'}$). From (2.12) it is clear that $g_s N$ then has to be large. In addition, the above descriptions have only taken into account tree level diagrams, so loop contributions should be suppressed – in other words, we need g_s small. All in all, we can do calculations on the string theory side (and trust the result) when

$$g_s N \gg 1, \qquad N \gg 1. \tag{2.16}$$

On the gauge theory side, the effective coupling is the 't Hooft coupling $\lambda = g_{YM}^2 N = 2\pi g_s N$. So the regime where the supergravity can be trusted corresponds to the strong 't Hooft coupling regime. The gauge theory on the other hand, is well understood only in the weak coupling. Hence, rather than a simple equivalence between two well-known theories, this is a strong/weak *duality*.

As "derived" here, this duality is between supergravity and gauge theory in the large N limit. And in this limit the correspondence is well established. However, the Maldacena conjecture (which is much harder to verify) states that it is valid for arbitrary N. It is this stronger form of the duality which usually goes under the name of the AdS/CFT correspondence.

Beyond the large N limit, the supergravity approximation is not valid anymore, and we have to work with the full string theory. But as mentioned already, how to do string theory calculations in the $AdS_5 \times S^5$ background is not well understood yet. One of the reasons that string theory on $AdS_5 \times S^5$ has remained an unsolved problem is the presence of RR flux.

Some reviews on the AdS/CFT correspondence are refs. [33–35].

2.2.2 Dictionary

The AdS/CFT correspondence says that string theory on $AdS_5 \times S^5$ is equivalent to a conformally invariant gauge theory in four flat dimensions with $\mathcal{N} = 4$ SUSY. One simple check of this correspondence is to compare the symmetries: The $SO(6)$ isometry group of S^5 is the R-symmetry group in the gauge theory, and $SO(4,2)$ isometry group of AdS_5 is the conformal group in four dimensions.

The correspondence relates partition functions of the two theories:

$$Z_{AdS}[\varphi^i] = Z_{CFT}[\varphi_0^i], \tag{2.17}$$

where the source φ_0^i of the appropriate operator $\mathcal{O}^i$ on the gauge theory side is identified with the boundary value of the supergravity field φ^i on the string theory side: $\varphi_0^i = \varphi^i(\rho = \infty)$. So fields in string theory are related to operators in the gauge theory. The details of this dictionary is not a priori clear, but it is possible to derive a relation between rank and mass m of fields, and conformal dimension Δ of the corresponding operator. For zero rank fields it is $2 + \sqrt{4 + m^2R^2} = \Delta$. The dilaton and the axion (RR zero-form field), for example, are zero rank massless fields that correspond to operators of dimension $\Delta = 4$, which turn out to be $\mathrm{Tr}(*F \wedge F)$ and $\mathrm{Tr}(F \wedge F)$ respectively.

Recall that with string theory in the $AdS_5 \times S^5$ background we really only have the supergravity limit available. The spectrum of string theory states is unknown, so we are unable to map operators to *string states*. This is a limitation of the AdS/CFT correspondence, and it would be very nice to find a string/gauge theory correspondence where we could go beyond the supergravity limit. This is exactly what the Berenstein-Maldacena-Nastase (BMN) limit does.

2.2.3 BMN limit

The discovery [28] of the BMN limit set off a lot of work on string theory in plane waves and its gauge theory dual. It is beyond the scope of this thesis to go into any detail on this duality, but since it does serve as motivation for what I will discuss in the next chapter, I will now very briefly summarise some of the basic observations.

In $AdS_5 \times S^5$ the energy is $E = i\partial_\tau$, and the angular momentum is $J = -i\partial_\psi$ (of a particle moving around the equator of S^5). In the gauge theory these translate into the conformal dimension Δ and the charge (also denoted J) of a $U(1)$ subgroup of the R-symmetry group. In the light-cone coordinates we get for the light-cone energy p_+ and momentum p_- (see eq. (2.6))

$$\begin{aligned} -p_+ &= i\frac{\partial}{\partial x^+} = \mu i\sqrt{2}(\partial_\tau + \partial_\psi) = \mu\sqrt{2}(\Delta - J), \\ p_- &= -i\frac{\partial}{\partial x^-} = -\frac{i\sqrt{2}}{\mu R^2}(\partial_\tau - \partial_\psi) = \frac{\sqrt{2}}{\mu}\frac{\Delta + J}{R^2}. \end{aligned} \tag{2.18}$$

Recall that $R^2 = \alpha'\sqrt{2g_{YM}^2 N} = \alpha'\sqrt{2\lambda}$, where $\lambda = g_{YM}^2 N$ is the 't Hooft coupling. The Penrose-Güven limit implies $R \to \infty$, meaning that both $N \to \infty$ and $\lambda \to \infty$.

Note that this is a *double* scaling limit (different from the 't Hooft limit where λ is kept finite).

The BMN limit is obtained by taking this limit while keeping energy and momenta of states finite in the string theory *i.e.*, we want $p_\pm$ finite. This means that $\Delta - J =$ finite, and $\Delta \sim J \sim R^2 \sim \sqrt{N}$. Only certain operators thus "survive" the scaling limit in the gauge theory, namely those for which

$$\Delta \sim J \sim \sqrt{N}, \qquad \Delta - J = \text{finite}. \tag{2.19}$$

The BMN limit therefore restricts to a particular large charge sector of the original $U(N)$ $\mathcal{N}=4$ supersymmetric gauge theory (2.15).

As already mentioned, the good thing is that we can now identify gauge theory operators with string theory states. For example, the ground state $|0;p_-\rangle$ has $\Delta - J = 0$, and there is a unique operator with this property, namely $\mathrm{Tr}[Z^J]$, where $Z = \frac{1}{\sqrt{2}}(\phi_5 + i\phi_6)$ is a complex scalar of dimension 1. (So Z^J has $\Delta = J$, and therefore $\Delta - J = 0$.) This dictionary can be extended to general string states.

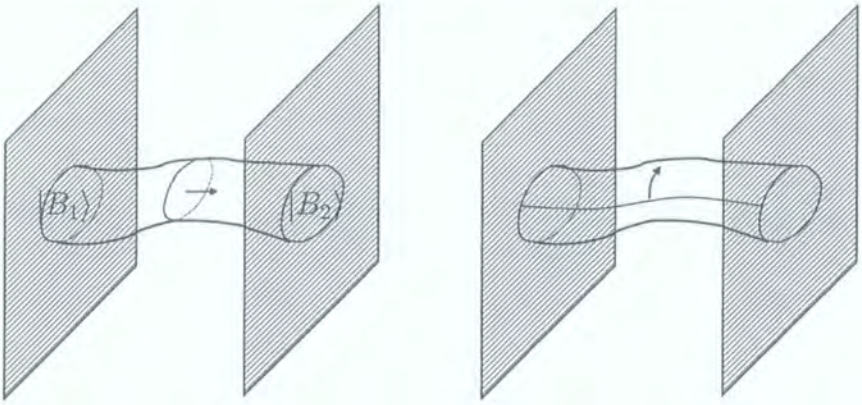
A more thorough review of the string/gauge theory correspondence in the BMN limit is ref. [36]. With all this motivation to study plane waves, let us now move on and do some calculations. The rest of this chapter is a review of work that has been done by others, and serves as introduction and background material for the next chapter.

2.3 D-brane interactions

This section serves as a preparation for next chapter on D-brane–anti-D-brane interactions in the plane wave background. Here, I will review some of the prerequisites. In particular, I will consider a general D-brane interaction represented by the cylinder diagram, and construct the amplitude for this diagram both in the closed string and open string pictures (see figure 2.1). The two descriptions are of course equivalent, telling us there must be an open/closed string duality.

2.3.1 D-branes in plane waves

As noted above, the plane wave background has an $SO(4) \times SO(4) \times \mathbb{Z}_2$ symmetry. It is therefore convenient [37, 38] to label D-branes in the plane wave background given in equation (2.1) as (r,s), if they are Euclidean, where r denotes the spatial extent in directions $i = 1,2,3,4$ (the first $SO(4)$) and s denotes the spatial extent in directions $i = 5,6,7,8$ (the second $SO(4)$). A Dp-brane then has $r + s = p + 1$.



(a) (b)

Figure 2.1: D-brane interaction viewed as exchange of a closed string (a), or as an open string vacuum loop (b).

If the D-branes are Lorentzian, then their worldvolume extends in the x^+ and x^- directions, and the notation is $(+, -, r, s)$. In that case, a Dp-brane has $r+s = p-1$.

The classification of possible D-branes in the plane wave background and various aspects of them have been studied in refs. [37–45]. First of all, there are two classes of D-branes. Define the matrix $\hat{M} = \prod_{m\in\mathcal{N}} \gamma^m$, where the product extends over Neumann directions. Then the product $\Pi\hat{M}\Pi\hat{M} = \pm 1$. The value of this product is what distinguishes the two classes of D-branes. *Class I* D-branes are the ones with $\Pi\hat{M}\Pi\hat{M} = +1$, and it turns out that these branes must have $|s-r| = 2$, and are confined to be at the origin in directions transverse to the brane. Only there do they correspond to supersymmetric solutions. *Class II* D-branes are the ones with $\Pi\hat{M}\Pi\hat{M} = -1$, and must have $|s-r| = 0, 4$, and can be at arbitrary position in the transverse directions. Table 2.1 summarises the possible D-brane configurations.

	Euclidean		Lorentzian	
class I	$p=1$	$(2,0)$, $(0,2)$	$p=3$	$(+,-,2,0)$, $(+,-,0,2)$
	$p=3$	$(3,1)$, $(1,3)$	$p=5$	$(+,-,3,1)$, $(+,-,1,3)$
	$p=5$	$(4,2)$, $(2,4)$	$p=7$	$(+,-,4,2)$, $(+,-,2,4)$
class II	$p=-1$	$(0,0)$	$p=1$	$(+,-,0,0)$
	$p=3$	$(0,4)$, $(4,0)$	$p=5$	$(+,-,0,4)$, $(+,-,4,0)$

Table 2.1: Summary of possible Dp-brane configurations in the plane wave background.

2.3.2 Interactions in flat space

Closed string picture

From the point of view of closed string theory, the cylinder diagram represents the exchange of a closed string between two Dp-branes. We write this amplitude

$$\mathcal{A}^{closed} = \langle B_2 | \Delta | B_1 \rangle, \tag{2.20}$$

where Δ is the closed string propagator, and $|B_1\rangle$ is the initial state and $|B_2\rangle$ is the final state for the closed string. These are *boundary states.*

Boundary states are defined by imposing boundary conditions as operator statements on closed string states. On the bosonic sector this means that the boundary state $|B\rangle$ is defined by

$$\begin{aligned} \partial_\tau X^m |B\rangle &= 0 \qquad (\mathrm{N}), \\ (X^i - Y^i)|B\rangle &= 0 \qquad (\mathrm{D}), \end{aligned} \tag{2.21}$$

where N denotes Neumann directions, and D denotes Dirichlet directions. The quantity Y^i denotes the position of the D-brane (or equivalently, the position of the closed string boundary state) in Dirichlet directions.

To derive explicit expressions for the boundary states, the procedure is to insert the closed string mode expansions for the operators X^μ, and then get equations in terms of modes. Solving these mode equations in the flat space case gives

$$|B_{flat}\rangle = \mathcal{N}_p \delta(X^i - Y^i) e^{-\sum_n \frac{1}{n}\left(\alpha^m_{-n}\tilde{\alpha}^m_{-n} - \alpha^i_{-n}\tilde{\alpha}^i_{-n}\right) + \text{fermions}} |0\rangle, \tag{2.22}$$

where $\mathcal{N}_p$ is a normalisation, and $|0\rangle$ denotes the ground state. The α's and $\tilde{\alpha}$'s are the closed string modes in the left-moving and right-moving sector respectively. I have denoted Neumann directions (parallel to the brane) by $m = 0, \ldots, p$, and Dirichlet directions (transverse to the brane) by $i = p+1, \ldots, D-1$. Notice the different signs in the exponent for Dirichlet directions and Neumann directions.

The propagator Δ is given by the Hamiltonian and can be written [11]:

$$\Delta \sim \int_0^\infty ds\ e^{-sH^{closed}}, \tag{2.23}$$

where the modular parameter s corresponds to the length of the cylinder. The infrared (IR) domain is for $s \to \infty$ and the ultraviolet (UV) is for $s \to 0$. When computing the amplitude (2.20), we also have to impose the level matching condition for closed strings.

Putting things together and performing the calculation, we get

$$\mathcal{A}^{closed} = \langle B|\Delta|B\rangle \sim \int_0^\infty \frac{ds}{2s} s^{\frac{p+1}{2}} e^{-\frac{z^2}{2\pi\alpha' s}} \frac{-f_4^8(\tilde{q}) + f_3^8(\tilde{q}) - f_2^8(\tilde{q})}{2s^4 f_1^8(\tilde{q})} = 0, \qquad (2.24)$$

where $\tilde{q} = e^{-2\pi s}$, and the last equality follows from the "abstruse identity" $-f_2^8 + f_3^8 - f_4^8 = 0$. The Jacobi f-functions are defined in appendix A.1. In the above expression, the $-f_4^8 + f_3^8$ terms are from the NSNS sector, whilst $-f_2^8$ comes from the RR sector.

The vanishing amplitude can be understood as resulting from a cancellation between contributions from the NSNS sector and the RR sector, $\mathcal{A} = \mathcal{A}_{NSNS} + \mathcal{A}_{RR} = 0$. The equal charge repulsion is cancelled by the gravitational attraction. This result means that the there is no force between the two D-branes, which should be no surprise as this system of two D-branes is a supersymmetry protected BPS state.

Open string picture

From the perspective of open strings, the cylinder diagram represents an open string vacuum loop *i.e.*, the partition function. It is very much like a vacuum loop in field theory, and we can use the Coleman-Weinberg formula to express it in terms of the open string Hamiltonian H^{open},

$$\mathcal{A}^{open} = \mathcal{Z} = \int_0^\infty \frac{dt}{2t} \mathrm{Tr} P_{GSO}^{+} e^{-tH^{open}}, \qquad (2.25)$$

where I have included the GSO projection operator $P_{GSO}^{+} = \frac{1+(-1)^F}{2}$ necessary in superstring theory. The trace has to be taken over all sectors, *i.e.* both the Ramond (R), and the Neveu-Schwarz (NS) sector. The modular parameter t represents the circumference of the loop, with the IR behaviour found at $t \to \infty$, and the UV at $t \to 0$. To compute this diagram, we need the explicit expression for the Hamiltonian.

In flat space the Hamiltonian is

$$H^{open} = \alpha'(p^m p_m + \frac{z^2}{4\pi^2\alpha'^2}) + \sum_n n N_n^{(b)} + \sum_r r N_r^{(f)} + a, \qquad (2.26)$$

where $a = 0$ in the R sector, and $a = -\frac{1}{2}$ in the NS sector. The z^2 term comes from the fact that the zero mode momentum is quantised in Dirichlet directions, $p^i = \frac{z^i}{2\pi\alpha'}$, where z^i is the separation between the branes. $N^{(b)}$ and $N^{(f)}$ are the boson and fermion number operators respectively. Inserting this Hamiltonian into eq. (2.25) gives

$$\mathcal{A}^{open} \sim \int_0^\infty \frac{dt}{2t} t^{-\frac{p+1}{2}} e^{-t\frac{z^2}{2\pi\alpha'}} \frac{-f_2^8(q) + f_3^8(q) - f_4^8(q)}{2f_1^8(q)} = 0, \tag{2.27}$$

where $q = e^{-2\pi t}$. The result that the amplitude vanishes is the same as the closed string calculation gave. This is of course as it should be. But we can also demonstrate the equivalence between the closed and open string descriptions more generally and instructively. This comes next.

Open/closed string duality

Starting with the closed string amplitude given in eq. (2.24), and substituting $s = \frac{1}{t}$, together with the identities

$$\begin{aligned} f_1(\tilde{q}) = \sqrt{t} f_1(q), \qquad f_2(\tilde{q}) = f_4(q), \qquad f_3(\tilde{q}) = f_3(q); \\ q = e^{-2\pi t}, \quad \tilde{q} = e^{-2\pi\frac{1}{t}}, \end{aligned} \tag{2.28}$$

gives

$$\mathcal{A}^{closed} \sim \int_0^\infty \frac{dt}{2t} t^{-\frac{p+1}{2}} e^{-\frac{z^2}{2\pi\alpha'}t} \frac{-f_2^8(q) + f_3^8(q) - f_4^8(q)}{2f_1^8(q)}. \tag{2.29}$$

This is the same as the open string amplitude in eq. (2.27), which confirms the open/closed duality,

$$\mathcal{A}^{closed} = \mathcal{A}^{open}. \tag{2.30}$$

Notice that this duality connects open string IR with closed string UV and vice versa.

2.3.3 Interactions in plane waves

As we have seen, string theory in plane waves is only simple if we choose a light-cone gauge. This has an immediate implication for the discussion of D-branes that will become very important in chapter 3.

The plane wave (2.1) that we have been discussing, is a solution of type IIB string theory, which is a theory of closed strings. It is therefore natural to use a light-cone gauge where we identify world-sheet time τ of the closed string with the spacetime light-cone direction x^+, exactly as we did in section 2.1.1,

$$\text{closed string light-cone gauge:} \qquad x^+ = \frac{M}{\mu}\tau, \tag{2.31}$$

with $M = 2\pi\alpha' p^+ \mu$. A consequence of this choice is that x^+ as well as x^- are

Dirichlet directions. That is, with this choice of gauge we can only discuss Euclidean (localised in time) D-branes of the type (r,s).

If we want to consider Lorentzian D-branes, *i.e.* D-branes extended in time, we can choose a light-cone gauge which is natural from the open string point of view, and identify *open string* world-sheet time $\tilde{\tau}$ with spacetime light-cone direction x^+,

$$\text{open string light-cone gauge:} \qquad x^+ = \frac{\tilde{m}}{\mu}\tilde{\tau}, \tag{2.32}$$

where $\tilde{m} = 2\pi\alpha'\tilde{p}^+\mu$ is the open string mass parameter, and $\tilde{p}^+$ is the + component of the open string light-cone momentum.

In the following I shall concentrate on Euclidean D-branes, following refs [38, 39, 41]. Lorentzian D-branes were first discussed in ref. [40].

Closed string picture

We already have the Hamiltonian for closed strings (in closed string light-cone gauge) in eq. (2.4). The other ingredient needed to compute the cylinder diagram are the boundary states. The general idea is the same as for boundary states in flat space, although the expressions involved are more complicated. The exact expression for the boundary state depends on what sort of D-brane we are dealing with, and I refer to refs. [38, 41] for a detailed treatment of the various possibilities.

As an illustrative example, and because it will be useful later, let me just write out the boundary state corresponding to a $(0,0)$-brane (or anti-brane). Define the coefficients $R_n^\pm$ and decompose the $SO(8)$ spinor modes S_n according to

$$\begin{aligned} R_n^\pm &= \sqrt{\frac{\omega_n \mp \eta M}{\omega_n \pm \eta M}}, \\ S_n^\pm &= \frac{1}{2}(1 \pm \Pi)S_n, \end{aligned} \tag{2.33}$$

with $\eta = +1$ for D-branes and $\eta = -1$ for anti-D-branes. Then the boundary state is:

$$\begin{aligned} |B\rangle &= |(0,0), x^i, \eta\rangle \\ &= (4\pi M)^2 \exp \sum_{k=1}^{\infty} \Big\{ \frac{1}{\omega_k} \alpha^i_{-k}\tilde{\alpha}^i_{-k} - i\eta R_k^+ S^+_{-k}\tilde{S}^+_{-k} - i\eta R_k^- S^-_{-k}\tilde{S}^-_{-k} |(0,0)\rangle_0 \Big\}, \end{aligned} \tag{2.34}$$

where the zero mode part is

$$|(0,0)\rangle_0 = |0\rangle_{ferm} e^{-\frac{Mx^2}{2}} e^{\frac{1}{2}a_0^i a_0^i - i\sqrt{2M}x^i a_0^i} |0\rangle_{bos}, \tag{2.35}$$

expressed in terms of the fermionic and bosonic part of the ground state. The parameter x^i is the position of the D-brane.

Let $z^i = x_2^i - x_1^i$, $z^\pm = x_2^\pm - x_1^\pm$ be the separation between the branes. The modular parameter s of the cylinder (length divided by circumference) is related to z^+ and p^+ via $s = \frac{z^+}{2\pi\alpha' p^+}$. Then the cylinder amplitude for interaction between a D$(0,0)$-brane and an anti-D$(0,0)$-brane can be written [38]

$$\mathcal{A}^{closed} = \int \frac{ds}{s} e^{-i\frac{z^+ z^-}{2\pi\alpha' s}} h_0^{(M)}(s;x_1,x_2) \frac{\left(g_2^{(M)}(\tilde{q})\right)^4}{\left(f_1^{(M)}(\tilde{q})\right)^8}, \tag{2.36}$$

where $\tilde{q} = e^{-2\pi s}$, and the zero mode contribution is

$$h_0^{(M)}(s;x_1,x_2) = \exp\Big\{-\frac{M(1+\tilde{q}^M)(x_1^2+x_2^2)}{2(1-\tilde{q}^M)} + \frac{2M\tilde{q}^{\frac{M}{2}} x_1 \cdot x_2}{(1-\tilde{q}^M)}\Big\}. \tag{2.37}$$

The f and g functions which appear in the oscillator part are defined in appendix A.1.

Note that M is not a free parameter in the integration (2.36), but is given in terms of s as $M = \frac{z^+\mu}{s}$. Note also that $\tilde{q}^M = e^{-2\pi z^+\mu}$ is independent of s.

Open string picture

Again, the cylinder diagram amplitude from the open string perspective is an open string vacuum loop, given as the trace over the exponentiated Hamiltonian, as in equation (2.25). However, to be consistent with the above, we need to work in the closed string light-cone gauge, which is a non-standard light-cone gauge from the open string point of view. Rather than relating the spacetime light-cone direction x^+ with open string worldsheet time $\tilde{\tau}$, it relates it to worldsheet *space* $\tilde{\sigma}$. Effectively, this interchanges the roles of z^+ and $2\pi\alpha' p^+$, and gives an open string mass parameter $m = \mu z^+$, which is different from $\tilde{m}$ in equation (2.32). The closed and open string mass parameters are related via $\frac{m}{M} = s$.

To get an expression for the Hamiltonian, it is necessary to solve the equations of motion with the appropriate boundary conditions for open strings. This has been done for various D-brane boundary conditions in ref. [38,41], and for open strings stretched between a $(0,0)$-brane and an anti-$(0,0)$-brane the result is

$$\frac{z^+}{2\pi} H^{open} = \frac{m}{2\sinh(\pi m)}\left(\cosh(\pi m)(x_2^i x_2^i + x_1^i x_2^i) - 2x_2^i x_1^i\right) + 2\pi \sum_{n\in\mathcal{P}_\pm} (\alpha_{-n}^i \alpha_n^i + |\omega_n| S_{-n}^a S_n^a), \tag{2.38}$$

where now, $\omega_n = \text{sign}(n)\sqrt{m^2+n^2}$, and the sets $\mathcal{P}_\pm$ are defined in appendix A.1.

This gives the amplitude

$$\mathcal{A}^{open} = \int_0^\infty \frac{dt}{2t} e^{it\frac{z^+z^-}{2\pi\alpha'}} \hat{h}_0^{(m)}(t;x_1,x_2) \frac{\hat{g}_4^{(m)}(q)^4}{f_1^{(m)}(q)^8}, \tag{2.39}$$

where $q = e^{-2\pi t}$, and

$$\hat{h}_0^{(m)}(t;x_1,x_2) = \exp\Big\{-\frac{mt}{2\alpha' \sinh(\pi m)}\big(\cosh(\pi m)(x_1^2+x_2^2) - 2x_1\cdot x_2\big)\Big\}, \tag{2.40}$$

and the $\hat{g}^{(m)}$ functions are defined in appendix A.1.

Open/closed duality

The open and closed string description of the cylinder diagram should be equivalent in the plane wave case just as in flat space, so we should expect eqns. (2.36) and (2.39) to be the same. And this is indeed the case, as follows from the modular transformation properties [38]

$$h_0^{(M)}(s;x_1^i,x_2^i) = \hat{h}_0^{(m)}(t;x_1^i,x_2^i), \qquad f_1^{(M)}(\tilde{q}) = f_1^{(m)}(\tilde{q}), \qquad g_2^{(M)}(\tilde{q}) = \tilde{g}_4^{(m)}(q), \tag{2.41}$$

where $t = \frac{1}{s}$, and $q = e^{-2\pi t}$, $\tilde{q} = e^{-2\pi s}$, and $m = Ms$.

I will continue the discussion of interactions between D-branes in the next chapter. Then I will use the ideas presented in this chapter and consider the interaction between D-branes and anti-D-branes. In particular, I will investigate an interesting stringy divergence in the amplitude which defines the so-called "stringy halo" of the D-branes.

But before that, I shall make a short digression on string thermodynamics and T-duality. This is somewhat detached from the focus of this and next chapter, but will connect to discussions in later chapters.

2.4 Hagedorn temperature and T-duality

The purpose of this section is mainly to introduce the concept of the Hagedorn temperature for strings in plane waves, and to relate it to T-duality. I have included it here as a link to later discussions on T-duality in null or timelike directions, and to the issue of closed timelike curves. But the subject is also very interesting on its own.

In summary, the Hagedorn temperature is the temperature in a thermal field theory above which the partition function diverges due to the density of states having an exponential growth that overcomes the Boltzmann suppression. In string theories it can alternatively be seen as the temperature where a closed string tachyon appears in the spectrum, signalling an instability of the theory.

An early and very interesting observation [46] was made about the Hagedorn temperature in thermal bosonic string theory: It is related to the self-T-dual radius of compactified Euclidean time. It therefore seems plausible that the study of T-duality can shed light on Hagedorn behaviour for strings in more general backgrounds. In light of this, it would be very interesting to see if we can make some connection between the Hagedorn temperature and T-duality in the plane wave case as well.

Thermal partition function

The plane wave spacetime is not static (manifestly not so by the presence of a $dtdx^9$ cross-term in the metric), and therefore a direct Euclideanisation of time gives a complex metric. It is therefore not clear that a thermodynamical theory defined by compactifying Euclidean time makes sense. However, the spacetime is still stationary, so a Hamiltonian approach should still work since it relies only on the existence of a timelike Killing vector.

In this section I am going to sketch the computation of the thermal partition function for a string, and demonstrate that it diverges above a certain temperature which is the Hagedorn temperature. The calculation presented here follows ref. [47].

In general, given a unit timelike Killing vector η, we can define a Hamiltonian by $H_\eta = \eta \cdot P$, where P is the canonical momentum operator. In particular, if we write $\eta = \frac{\xi}{|\xi|}$, we get $\beta H_\eta = \xi \cdot P$, where $\beta = |\xi|$ is the inverse temperature. The normal Hamiltonian is defined by the Killing vector $\xi = \partial_\tau$, and the light-cone Hamiltonian is defined by the Killing vector $\xi = \partial_+$.

Let us work in light-cone coordinates $x^+ = 2\pi\alpha' p^+ \tau$, and choose a Killing vector $\xi = a\partial_+ - b\partial_-$.

The thermal one-string partition function is

$$Z_1 = \mathrm{Tr}\; e^{-\beta H}\delta(\mathcal{P}) = \mathrm{Tr}\; e^{bP_- - aP_+}\delta(\mathcal{P}), \tag{2.42}$$

where $\mathcal{P} \equiv \sum_{n>0} n(N_n - \tilde{N}_n) = 0$ is a level-matching constraint which is imposed by the conformal symmetry upon choosing light-cone gauge. It is also convenient to define the light-cone one-string partition function,

$$z_{lc}(\tau_1, \tau_2) = \text{Tr}\; e^{-2\pi\tau_2 H + 2\pi i \tau_1 \mathcal{P}}. \tag{2.43}$$

The momentum component P_- of the string is determined by the light-cone gauge to be $P_- = p^+$. The component P_+ is the light-cone Hamiltonian, and is related to the usual Hamiltonian via $P_+ = H_{lc} = \frac{1}{2\pi\alpha' p^+}H$. Note that the trace involves an integral over P_-.

In order to express the partition function (2.42) in terms of the light-cone partition function (2.43), we write the delta function as an integral over τ_1, and make the change of variables $p^+ \longrightarrow \tau_2$ defined by $\tau_2 = -\frac{a}{2\pi\alpha' p^+}$. Doing this we find

$$Z_1 = \frac{a}{2\pi\alpha'} \int \frac{d\tau_1 d\tau_2}{\tau_2^2} e^{-\frac{ab}{2\pi\alpha'\tau_2}} z_{lc}(\tau_1, \tau_2) \tag{2.44}$$

To determine the behaviour of the partition function, we therefore need the explicit expression for z_{lc}, which again depends on the Hamiltonian. I will not dwell on the details here, and just state the results of the calculations:

In *Minkowski space* the temperature T is related to a and b via $T^{-1} = \beta = \sqrt{-\xi \cdot \xi} = \sqrt{2ab}$. It turns out that

$$z_{lc}(\tau_1, \tau_2) = \tau_2^{-4} \Big(\prod_{n=1}^{\infty} \frac{(1+q^n)(1+\bar{q}^n)}{(1-q^n)(1-\bar{q}^n)} \Big)^8 \overset{\tau_2 \to 0}{\sim} \tau_2^4 e^{\frac{4\pi}{\tau_2}}, \tag{2.45}$$

where $q = e^{2\pi i \tau}$, $\tau = \tau_1 + i\tau_2$, and the numerator and denominator within the product come from fermionic and bosonic oscillator modes respectively. The prefactor t_2^{-4} is due to the zero modes. For small τ_2 the integrand in eq. (2.44) therefore behaves like

$$\tau_2^2 e^{\frac{1}{\tau_2}\left(-\frac{ab}{2\pi\alpha'} + 4\pi\right)}, \tag{2.46}$$

which means that the partition function diverges (in the $\tau_2 \longrightarrow 0$ domain) if

$$2ab < 16\pi^2\alpha' \equiv \beta_H^2. \tag{2.47}$$

The limiting temperature $T_H = \beta_H^{-1}$ is the Hagedorn temperature.

In the *plane wave space*, the temperature is given by $T^{-1} = \beta = \sqrt{-\mu^2 x^2 a^2 + 2ab}$, and [47]

$$z_{lc}(\tau_1, \tau_2) = \left(\frac{Z^{(M)}_{1/2,0}(\tau_1, \tau_2)}{Z^{(M)}_{0,0}(\tau_1, \tau_2)} \right)^4 \overset{\tau_2 \to 0}{\sim} \tau_2^{-2} e^{\frac{B(a)}{\tau_2}}, \tag{2.48}$$

where $M = 2\pi\alpha' p^+ \mu$, and $B(a)$ is some factor that is independent of τ_1 and τ_2, but does depend on a. The "deformed theta functions" $Z^{(M)}_{1/2,0}$ and $Z^{(M)}_{0,0}$ are defined in appendix A.2. The partition function now diverges if

$$2ab < 4\pi\alpha' B(a). \tag{2.49}$$

The values of a and b which saturate this inequality define the Hagedorn temperature in the plane wave case. Note that the temperature depends on position x^i, which is a redshift effect.

Closed string tachyon and T-duality

In Minkowski space it is also possible to study thermodynamics by Wick rotating time, and make Euclidean time compact. The circumference of this compact direction is then identified with the inverse temperature β.

If we consider string theory in this setting, there is the possibility of winding in the Euclidean time direction, which changes the spectrum from the usual Minkowski case. It is straightforward to work out, and for the bosonic case the result is

$$\begin{aligned} N - \tilde{N} &= nw, \\ M^2 &= \frac{4}{\alpha'^2}\Big(\big(\frac{\alpha'\pi}{\beta}n\big)^2 + \big(\frac{\beta}{4\pi}w\big)^2\Big) + \frac{2}{\alpha'}(N + \tilde{N} - 2), \end{aligned} \tag{2.50}$$

where w is the winding number, n is the momentum number, and N and $\tilde{N}$ are the number operators in the left- and right-moving sectors respectively. The negative mass squared for $n = w = N = \tilde{N} = 0$ is the usual tachyon in bosonic theory that is removed by the GSO projection in superstring theory. So we shall not worry about this tachyon.

Consider instead $w = 1$, $n = N = \tilde{N} = 0$, which gives

$$M^2 = \frac{4}{\alpha'}\Big(\frac{\beta^2}{16\pi^2\alpha'} - 1\Big). \tag{2.51}$$

This state becomes tachyonic if

$$\beta^2 < 16\pi^2\alpha' \equiv \beta_H, \tag{2.52}$$

that is, if the temperature is above the Hagedorn temperature. And, importantly, this state is *not* removed by the GSO projection in superstring theory. So this result derived for the bosonic string, is also true for the superstring.

So, indeed, the Hagedorn temperature is the temperature at which a new tachyon appears in the spectrum, making the partition function divergent, and telling us that the vacuum we expand around is not the true vacuum anymore.

It was noted already in the classic paper [46] that the Hagedorn temperature is related to the radius of self-duality under a transformation that is now well-known as T-duality.

Consider again the bosonic string in Minkowski space, where one direction is compactified with a radius R. A T-duality transformation along this direction gives the same theory back, but with a new radius for the compact direction, $R \to R' = \frac{\alpha'}{R}$. For the particular value $R_{sd} = \sqrt{\alpha'}$ we see that the theory is self-dual. And if this compact direction is the Euclidean time direction, we get that the self-dual inverse temperature is $\beta_{sd} = 2\pi R = 2\pi\sqrt{\alpha'}$. This is also apparent from the mass spectrum (2.50), if we accompany the change in radius with an interchange of momentum number n and winding number w.

In conclusion, the inverse Hagedorn temperature in Minkowski space is two times the self-T-dual temperature, $\beta_H = 2\beta_{sd}$.

This could of course be just a coincidence valid only for string theory in Minkowski space. But it is also not unreasonable to think that T-duality has a lot to teach us concerning the Hagedorn temperature in general backgrounds as well. At least it would be very interesting to generalise these calculations to other string theory models where feasible, and see explicitly what we get.

Turning then to the plane wave solution, we are immediately faced with a fundamental problem. In working with string theory in plane waves, we use light-cone gauge, which identifies time with a (timelike) light-cone direction x^+. So it seems that a T-duality understanding of the Hagedorn point in this case would have to involve T-duality in a timelike direction. Whether it is possible to make sense of T-dualities in timelike, or even null directions is an interesting question which I am going to return to in chapter 4.

The question of how much the study of T-dualities can teach us about the Hagedorn behaviour, or whether the flat space result can be extended to other backgrounds at all, is an interesting one which I have no further comments on at the moment. I have included it here because the connection is worth keeping in mind, and also because it motivates the investigations in chapter 4.

Chapter 3

D-brane–anti-D-brane interactions in plane waves

In this chapter I study aspects of the interaction between a D-brane and an anti-D-brane in the maximally supersymmetric plane wave background of type IIB superstring theory (2.1), which is equipped with a mass parameter μ. An early such study in flat spacetime ($\mu = 0$) served to sharpen intuition about D-brane interactions, showing in particular the key role of the "stringy halo" that surrounds a D-brane. The halo marks the edge of the region within which tachyon condensation occurs, opening a gateway to new non-trivial vacua of the theory. To learn more both about the plane wave background, and about D-branes in general, it is interesting to study the fate of the halo for non-zero μ. I shall focus on the simplest cases of a Lorentzian brane with $p = 1$ and a Euclidean brane with $p = -1$, the D-instanton. For the Lorentzian brane, the halo turns out to be unaffected by the presence of non-zero μ. This most likely extends to other (Lorentzian) p. For the Euclidean brane, on the other hand, we will see that the halo *is* affected by non-zero μ. As this is related to subtleties in defining the exchange amplitude between Euclidean branes in the open string sector, I expect this to extend to all Euclidean branes in this background.

The results of this chapter have been published as ref. [1] with my supervisor Prof. Clifford V. Johnson as co-author.

3.1 Introduction

A D-brane and its "anti-particle", an anti-D-brane, upon approaching each other, will annihilate. The generic product of this annihilation process is expected to be a state of closed strings, which carry no net RR charge. This expectation is supported by field theory intuition and knowledge of which objects are the carriers of the available conserved charges in perturbative string theory. From experience with field theory we would expect to be able to see the beginnings of the process of annihilation via the opening up of new decay channels at coincidence. These can be seen by studying the amplitude for exchange of quanta between the two branes, which gives a potential. At small separations, the behaviour of the interaction potential can signal new physics. Basically, a divergence in the amplitude as the objects are brought together can signal the opening up of a new channel (or new channels) not included in the computation of the amplitude away from the divergent regime.

In field theory, for a separation z of the two objects, the divergence follows simply from the fact that the amplitude for exchange is controlled by the position space propagator $\Delta(z)$ which (for more than two transverse directions) is divergent at $z = 0$. This is where the new channels can open up, which can include the processes for complete annihilation into a new sector, if permitted by the symmetries of the theory.

3.1.1 Superstrings in flat space

For D-branes in superstring theory, such a divergence does indeed show up, but there is an important new feature [48]. The divergence occurs when the D-branes are finitely separated, by an amount set by $z_h^2 = 2\pi^2\alpha'$, where α' is the characteristic length scale set by a fundamental string's tension. This is interpreted as the fact that in addition to the many special features of D-branes, they have a "stringy halo" originating in the fact that the bulk of the open strings which (by definition) end on them can reach out in the transverse directions, forming a region of potential activity of size set by z_h. This halo means that the D–branes can interact with each other before zero separation, as there is an enhancement of the physics of interaction by new light states formed by the entanglement of the halos, and the crossover into the annihilation channel begins before the branes are coincident.

Now, let me show how to compute the size z_h of the halo. Anti-D-branes are D-branes with opposite orientation, which means they have the opposite RR charge, and therefore that the RR contribution to the amplitude comes with the opposite sign as compared to D-branes. So in the case of a D-brane–anti-D-brane interaction, the NSNS and RR contributions add up rather than cancel:

$$\begin{aligned} \mathrm{D}p\text{–}\mathrm{D}p: &\quad \mathcal{A} = \mathcal{A}_{NSNS} + \mathcal{A}_{RR} = 0, \\ \mathrm{D}p\text{–}\overline{\mathrm{D}p}: &\quad \mathcal{A} = \mathcal{A}_{NSNS} - \mathcal{A}_{RR} = -2\mathcal{A}_{RR}. \end{aligned} \tag{3.1}$$

From equations (2.24) and (2.28) we therefore get (after a modular transformation) for the D-brane–anti-D-brane cylinder amplitude

$$\begin{aligned}\mathcal{A} = -2\mathcal{A}_{RR} &\sim \int_0^\infty \frac{dt}{t} t^{-\frac{p+1}{2}} e^{-\frac{z^2}{2\pi\alpha'}t} \frac{f_4^8(q)}{f_1^8(q)} \\ &\sim \int_0^\infty \frac{dt}{t} t^{-\frac{p+1}{2}} e^{-\pi t\left(\frac{z^2}{2\pi^2\alpha'}-1\right)} G(t),\end{aligned} \tag{3.2}$$

where $q^{-2\pi t}$, and $G(t) = q^{\frac{1}{2}} \frac{f_4^8(q)}{f_1^8(q)}$ is a bounded function whose explicit form is not important for the present discussion.

It is apparent from the above that the force $F = -\frac{\partial \mathcal{A}}{\partial z}$ between the brane and anti-brane diverges in the open string IR ($t \to \infty$) if $\frac{z^2}{2\pi^2\alpha'} - 1 < 0$, *i.e.* if

$$z^2 < 2\pi^2\alpha' \equiv z_h^2. \tag{3.3}$$

This defines the size z_h of the stringy halo.

Recall that the amplitude of exchange can be thought of using two equivalent pictures: Either as tree level exchange of closed string quanta between the branes, or (after a modular transformation) as the one-loop vacuum diagram for open strings stretched between the two D-branes. In the open string description, at separation z_h, the lightest open string becomes massless, and for any closer separation it becomes tachyonic, signalling that the entire vacuum configuration is unstable and wishes to roll to another vacuum. It is this tachyon which produces the divergence in the amplitude, converting a decaying exponential into a growing one, spoiling the convergence of the amplitude in the infrared (IR) region.

It is easy to see how this happens. In flat space, in the usual RNS formulation of string theory, the worldsheet Hamiltonian is given as $H = L_0 = \alpha' p^2 + N + a_{R(NS)}$, where the constant $a_{R(NS)}$ is the zero point energy and N is the total number operator. The zero point energy is $a_R = 0$ in the Ramond sector (R), and $a_{NS} = -\frac{1}{2}$ in the Neveu-Schwarz (NS) sector.

For strings stretched between two D-branes, we have $p^m = x^m/2\pi\alpha'$ for transverse (to the branes) directions x^m. So, splitting transverse (labelled m) and parallel (labelled i) directions we can write

$$L_0 = \alpha' p^i p_i + N + \frac{z^2}{4\pi^2\alpha'} + a_{R(NS)} \; . \tag{3.4}$$

This gives a mass spectrum

$$M^2 = -p^i p_i = \frac{1}{\alpha'}\left(N + a_{R(NS)} + \frac{z^2}{4\pi^2\alpha'}\right) \; . \tag{3.5}$$

The NS ground state ($N = 0$, $a_{NS} = -\frac{1}{2}$) has mass squared

$$M_0^2 = \frac{1}{2\alpha'}\left(\frac{z^2}{2\pi^2\alpha'} - 1\right) . \tag{3.6}$$

This is a tachyon if $z^2 < 2\pi^2\alpha'$.

In the usual case this ground state is eliminated by the GSO projection $P^+ = \frac{1+(-1)^F}{2}$ in superstring theory. But when we consider a brane-anti-brane system, we are effectively reversing the GSO projection in the partition function, giving $P^+ \to P^- = \frac{1-(-1)^F}{2}$, since anti-branes come with a minus sign. This means that the NS ground state ($N = 0$) will now survive, and the possible tachyon above is present in the spectrum. So for $z^2 < 2\pi^2\alpha'$ there is a tachyon, and so there is a one-to-one correspondence between the tachyon's appearance and divergence of the integral. (For the case when all of the directions are transverse, as is the case for D-instantons, the tachyon interpretation follows from continuation and T-duality.)

Note that this is an *open* string tachyon, and has nothing to do with the *closed* string tachyon which we encountered in section 2.4 as an instability at the Hagedorn temperature.

The D-branes annihilate via conversion to closed strings in the generic situation, but the tachyon picture can be exploited in a very nice way to produce more structure [49–52]. For the $G = U(N) \times U(N)$ gauge theory on the $(p+1)$-dimensional world-volume on N Dp-branes and N anti-Dp-branes, the tachyon field, transforming as the $(\mathbf{N}, \bar{\mathbf{N}})$, can be put into a configuration endowed with non-trivial topological charge, and the tachyon potential needs not yield a runaway to a sector containing only closed strings. Having such topological vacuum solutions in the tachyon sector allows for the possibility of a stable remnant –interpreted as a D-brane of lower dimension– of the annihilation process after the "dust" made up by closed string products has cleared. It turns out that the spectrum of hypermultiplets in the $U(N) \times U(N)$ world-volume theory supplies a set of variables which is isomorphic to those needed to perform a K-theoretic analysis of the topology of G-vector bundles over the worldvolume. And so the classification of all D-branes which can appear on a spacetime is apparently elegantly and economically obtained by using the results of the appropriate K-theory of the spacetime which the Dp-branes and anti-Dp-branes fill [53–55]. The case of $p = 9$ for Minkowski spacetime yields the entire classification of D-branes in the most familiar symmetric vacuum of type IIB superstring theory.

3.1.2 Plane waves

This is all well understood for the case of flat ten dimensional spacetime. So when we encounter another background which has the same maximal supersymmetry as flat spacetime, namely the the plane wave with RR flux as given in eq. (2.1), questions naturally spring to mind about the key lessons we have learned about D-branes. Is the picture of D-branes as Dirichlet open string boundary conditions as powerful in this context as it has been in flat spacetime? In particular, do the dynamics hidden within the stringy halo of the branes bear any similarity to the flat spacetime case? Are all D-branes classified by K-theory, now of the new background?

This chapter shows that the properties of the halo –the fact that it exists, and also its location and size– are unaffected by non-zero μ for all branes that have a Lorentzian definition, *i.e.*, are at a definite position in space, but not time. So this particular (and important) property of D-branes in this non-trivial RR background is very much like that in flat space. This bodes well for an attempt to classify such D-branes in this background using tachyon condensation and K-theory. However, for branes with a Euclidean definition, such as the $p = -1$ brane, we will find that the halo –or at least its analogue in this context– is deformed by non-zero μ.

3.2 The Interaction

The string theory diagram of interest is a cylinder, representing either the tree level exchange of closed string quanta between two D-branes, or the one-loop vacuum process involving the circulation of open strings with ends on either D-brane. See figure 3.1.

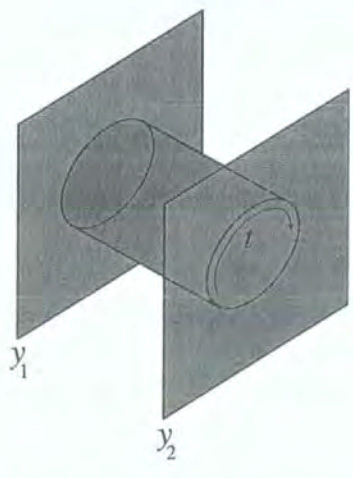


Figure 3.1: Cylinder diagram for computing the amplitude of interaction between two branes. The parameter t is open string propagation time, and is the modulus of the cylinder.

I will focus on the results for the simplest branes in the Euclidean and Lorentzian classes. These are the D(-1)-branes (or $(0,0)$-branes), and the D1–branes (or $(+,-,0,0)$-branes), discussed in ref. [38]. The former requires the time direction, in which the branes are also pointlike, to be Euclidean.

The results are reasonably simple for these cases, compared to other (r,s) with $r \neq s \neq 0$, and it would be interesting to explore those other cases in detail. I expect that the key observations made here for these $r = 0 = s$ cases will be quite generic, although there may be additional features to be deduced from studying other cases in detail.

3.2.1 The Amplitude and Potential

Consider a Dp-brane and its antiparticle for $p = \pm 1$. If $p = -1$, it is an instanton, and we consider it to be pointlike in Euclidean time. If $p = +1$ it is a string, and the theory is Lorentzian.

Place a Dp-brane at position y_1^i in the x^i directions ($i = 1, \dots, 8$), and a $\overline{\mathrm{D}p}$-brane (anti-brane) at position y_2^i, with a separation $z^\pm$ in the $x^\pm$ directions if $p = -1$. The details of the derivation of the amplitudes can be found in ref. [38], and since we will not need them all here, I refer the reader there for more information.

We want the amplitude in terms of the open string channel, because that is where it is easy to analyse the domain (open string IR) where the divergence appears.

For the case $p = 1$, this can be done directly with open string light cone gauge (2.32): $x^+ = 2\pi\alpha'\tilde{p}^+\tilde{\tau}$. So we have

$$t = \frac{z^+}{2\pi\alpha'\tilde{p}^+}\ , \quad m = 2\pi\alpha'\tilde{p}^+\mu \tag{3.7}$$

where $\tilde{p}^+$ is the open string momentum component in the x^+ direction. I denoted the mass parameter $\tilde{m}$ rather than m in the previous chapter, but I change this notation so that I can treat the $p = 1$ and $p = -1$ brane together in the following.

For the case $p = -1$, however, things are more subtle. A Dirichlet condition is needed in the time direction, but this is incompatible with the open string light-cone gauge choice. Instead, The amplitude is defined in the closed string sector by propagating for a distance z^+ between two boundary states. The propagation time is

$$s = \frac{z^+}{2\pi\alpha' p^+}\ , \tag{3.8}$$

since closed string light-cone gauge is $x^+ = 2\pi\alpha' p^+ \tau$, with mass parameter

$$M = 2\pi\alpha' p^+ \mu \ . \tag{3.9}$$

The open-closed string duality is then invoked to define the amplitude in terms of the open string channel. We have in fact already written down this cylinder amplitude in equation. (2.39). The modular transformation from closed to open strings means

$$t = 1/s = \frac{2\pi\alpha' p^+}{z^+} \ , \quad \text{and} \quad m = \mu z^+ = M t^{-1} \ . \tag{3.10}$$

This will be very important later.

Introduce a parameter ϵ, and let $\epsilon = 0$ for the $p = 1$ case, and $\epsilon = 1$ for the $p = -1$ case. Then we can summarise the amplitude as

$$\mathcal{A} = \int_0^\infty \frac{dt}{2t} t^{-(1-\epsilon)} e^{i\epsilon \frac{z^+ z^-}{2\pi\alpha'} t} \hat{h}_0(t; y_1, y_2) \frac{\hat{g}_4^{(m)}(q)^4}{f_1^{(m)}(q)^8} \ . \tag{3.11}$$

where $q = e^{-2\pi t}$, and a double Wick rotation has been performed: $\tau \to it$, $x^+ \to ix^+$. It is important to note that the meaning of m in this expression is different for the two cases. The functions involved in the above expressions are defined in appendix A.1.

For higher p, (which I will not be considering here) there are no additional powers of t in the integrand. These are normally due to integration over continuous zero modes in the flat spacetime case. The plane wave background has no such modes for the directions x^i, (the zero modes are instead themselves harmonic oscillators [56–59]) and so no such t^{-1} factors beyond those appearing here are present.

3.2.2 Divergences, Tachyons, and the Halo

What is important for our discussion is the structure of the full amplitude for the cylinder diagram, given above in equation (3.11) as an integral over the modulus t. It can be written as:

$$\mathcal{A} = \int_0^\infty \frac{dt}{2t} t^{-(1-\epsilon)} \exp\Big\{-2\pi t Z(m, y_1, y_2)\Big\} G(t), \tag{3.12}$$

where the exponent $Z(m, y_1, y_2))$ is defined as

$$\begin{aligned} Z(m, y_1, y_2) =& \frac{m\pi}{4\pi^2\alpha' \sinh(m\pi)} \left[\cosh(m\pi)(y_1^2 + y_2^2) - 2y_1 \cdot y_2\right] \\ & - 4(\hat{\Delta}_m - 2\Delta_m) - i\epsilon \frac{z^+ z^-}{4\pi^2\alpha'} \ , \end{aligned} \tag{3.13}$$

and the function $G(t)$ is defined as:

$$\begin{aligned} G(t) &= \frac{\prod_{l\in\mathcal{P}_+}(1-q^{|\omega_l|})^2\prod_{l\in\mathcal{P}_-}(1-q^{|\omega_l|})^2}{(1-q^m)^4\prod_{n=1}^{\infty}(1-q^{\omega_n})^8} \\ &= \prod_{n=1}^{\infty}(1-q^{\omega_n})^{-8}\prod_{l\in\mathcal{P}_+,l>0}(1-q^{\omega_l})^4\prod_{l\in\mathcal{P}_-,l>0}(1-q^{\omega_l})^4\ , \end{aligned} \tag{3.14}$$

where $\omega_l = \mathrm{sign}(l)\sqrt{l^2+m^2}$ and the sets $\mathcal{P}_\pm$ are defined in appendix A.1. For our discussion, the only important fact about the function $G(t)$ is that its behaviour at large and small t is such that generically, the amplitude is convergent. That $\mathcal{A}$ is finite as $t\to 0$ follows from the fact that small t is the closed string IR limit, where this amplitude should reproduce simple low energy field theory results for massless exchange at tree level. The $t\to\infty$ limit is also well behaved generically, since this is the open string IR limit, which is fine – away from special circumstances which will not show up in the oscillator contributions since their energies are higher than the lowest lying states. In fact, it is clear that $G(t)\to 1$ as $t\to\infty$, and so whether A is finite as $t\to\infty$ depends on the sign of the exponent Z, which controls those lowest lying states.

The divergence for negative Z is related to the lowest lying states becoming tachyonic at this point, as is most familiar in the RNS formulation in the flat spacetime background (see section 3.1.1.)

Let us write everything in terms z^i, the separation between the branes in the eight directions x^i, defined by $y_2^i = y_1^i + z^i$. The expression for Z then becomes (recall $\epsilon = 0$ for $p=1$ case, and $\epsilon = 1$ for $p=-1$ case)

$$Z(m,y_1,z) = \frac{1}{4\pi^2\alpha'}\frac{m\pi}{\tanh(m\pi)}\Big[(z+a)^2 - i\epsilon\frac{\tanh(m\pi)}{m\pi}z^+z^- - b^2\Big], \tag{3.15}$$

where I have defined:

$$a = \frac{\cosh(m\pi)-1}{\cosh(m\pi)}y_1\ ,\quad b=\tanh(m\pi)\sqrt{y_*^2-y_1^2}\ ,\quad y_*^2 = \frac{16\pi^2\alpha'(\hat{\Delta}_m - 2\Delta_m)}{m\pi\tanh(m\pi)}\ . \tag{3.16}$$

For the Lorentzian $p=1$ case, these parameters simplify further in the $t\to\infty$ limit of interest. Since for fixed z^+ the large t region corresponds to small $\tilde{p}^+$ (this follows from equation (3.7), or on general grounds from the operator definition of the amplitude), we see that $m \longrightarrow 0$ in all of these expressions, and so we obtain:

$$Z \longrightarrow \frac{1}{4\pi^2\alpha'}(z^2 - 2\pi^2\alpha') \ , \tag{3.17}$$

This is in fact the same expression we would obtain from the equivalent flat space computation, which simply has $m = 0$ throughout, and so we recover the well known [48] divergence at separation given by $z_h^2 = 2\pi^2\alpha'$. In fact, the result ought to be present for all Lorentzian branes, as the relevant amplitude can be defined directly in the open string light cone gauge. Intuitively, we are looking for a result in the open string IR limit $t \longrightarrow \infty$, which (from equation (3.7)) corresponds to $\tilde{p}^+ \longrightarrow 0$. But the parameter upon which any new physics can depend is $m = 2\pi\alpha'\tilde{p}^+\mu$, which vanishes in the limit. So there is no new physics.

For the Euclidean $p = -1$ case, the situation is very different. Now, for a given separation z^+, the $t \longrightarrow \infty$ limit corresponds (due to equation (3.10)) to $p^+ \longrightarrow \infty$ (this is the *closed string* momentum) and so things get quite reversed. In fact, the natural mass parameter seen by the open string physics is $m = \mu z^+$. There is therefore quite a complicated dependence on z^+, as is evident from the equation (3.15). Looking (without loss of generality, since the spacetime is homogeneous) at the case where we put one brane at the origin in the transverse directions, and so $y_1^i = 0$ and $z^i = y_2^i$, then the vanishing of Z can be written:

$$z^2 - i\frac{\tanh(\pi\mu z^+)}{\pi\mu z^+}z^+z^- = 2\pi^2\alpha'\mathcal{D}(\mu z^+)\frac{\tanh(\pi\mu z^+)}{\pi\mu z^+} \ , \tag{3.18}$$

where

$$\mathcal{D}(\mu z^+) = 8\left(\hat{\Delta}_m - 2\Delta_m\right) \ . \tag{3.19}$$

Note that Δ_m and $\hat{\Delta}_m$ are zero-point energies which arise naturally in the closed and open string sectors, respectively. They behave like $\Delta_m \longrightarrow -1/48$ and $\hat{\Delta}_m \longrightarrow 1/12$ when $m = \mu z^+$ tends to zero [38]. The quantity $\mathcal{D}(\mu z^+)$ decreases from unity and asymptotes to zero as μz^+ increases. Of course, when μ (and hence m) vanishes, this gives the expected result:

$$z^2 - iz^+z^- = 2\pi^2\alpha' \equiv z_h^2 \ . \tag{3.20}$$

Note here that the unusual factor of $-i$ in this expression is as a result of the Wick rotation, which results in the (complexified) metric

$$ds^2 = -2idx^+dx^- + \mu^2x^2(dx^+)^2 + \sum_{i=1}^{8} dx^idx^i \ . \tag{3.21}$$

For non-zero μ it is hard to interpret the result cleanly, but there is certainly a non-trivial dependence of the location of the "halo" on μ, in contrast to the Lorentzian case.

As a simple special case, we can place the branes at the same transverse position, and hence $z^i = 0$. Then we have the equation:

$$-iz^+z^- = 16\pi^2\alpha' \left(\hat{\Delta}_m - 2\Delta_m\right) . \tag{3.22}$$

For orientation, let us consider the flat space case $\mu = 0$. We can continue to a more familiar Lorentzian picture by choosing $z^- \to iz^-$. This gives a hyperbola in the plane, with equation

$$z^+z^- = 2\pi^2\alpha' \equiv z_h^2 . \tag{3.23}$$

Contrast this to the case of field theory, where the right hand side would be zero, giving us the light-cone. This is as expected for point-like behaviour. The flat space string theory result gives us a hyperbola. This is the manifestation of the halo which broadens out the available region of contact by widening the light-cone into a sort of "light-funnel". For the $\mu \neq 0$ case, the hyperbola is deformed, since z^- decreases more rapidly with increasing z^+ than before due to the behaviour of the function $\mathcal{D}(\mu z^+)$ discussed below equation (3.19). See figure 3.2.

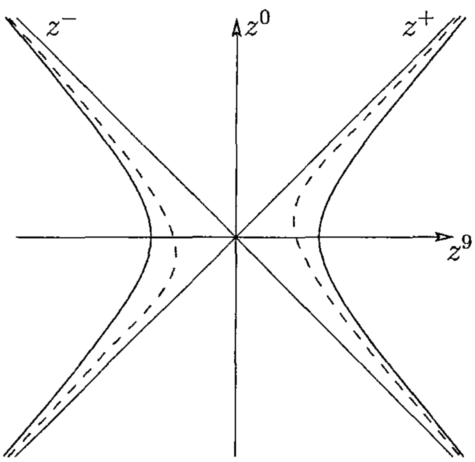


Figure 3.2: The hyperbola (solid curve) represents the edge of the "halo" for D-instantons in flat space, $\mu = 0$. For the $\mu \neq 0$ case, it is deformed to the dashed curve. The field theory result is the pair of lines $z^+z^- = 0$.

For the interpretation of the shape of the halo for non-zero μ once the transverse positions of the branes are different from each other, more work is needed. This is because the metric is no longer flat, and furthermore, we have to take seriously the matter of the Euclidean continuation of the metric implied in the computation of the amplitude. The choices made mean that the metric is no longer real (see equation (3.21)), and this presents difficulties of interpretation which must be explored further.

3.3 Summary

We have seen that the structure of the halo for Lorentzian branes in the plane wave background is independent of μ, giving the same physics as for D-branes in flat space. This is because the mass parameter induced by non-zero μ in the effective world-volume theory vanishes in the open string IR limit, the regime where the halo is to be found. For the D-instanton (and presumably all Euclidean branes) this is not the case, since their being pointlike in the $x^{\pm}$ directions requires the relevant amplitudes to be defined by starting with the closed string light cone gauge and then arriving at the open string physics by duality. The resulting open string physics sees a mass parameter which does not vanish in the IR limit, and hence the physics of the halo is not the same as in flat space. The significance of this non-trivial μ dependence of the structure of the halo of the D-instanton (and by extension, all Euclidean branes defined by starting with the closed string amplitude) is not clear at present. However, it may have some significance, since D-instantons contribute to type IIB string theory processes non-perturbatively (see *e.g.*, ref. [60]). See also ref. [61] for another study of D-brane interactions in plane wave backgrounds where the difference in Euclidean and Lorentzian branes is discussed.

Chapter 4

T-dualities in null directions, and closed timelike curves

This chapter is a detour from the general theme of exact string theory solutions, but nevertheless serves as a motivational link between the study of plane waves in previous chapters, and the study of stringy Taub-NUT spacetime in the next chapters. The central issues are T-dualities in null/timelike directions and closed timelike curves, which I will address by studying a rotating D1-D5-pp supergravity solution.

We saw in section 2.4 that the Hagedorn temperature of thermal string theory is related to the self-dual radius under a T-duality transformation. And in order to carry this correspondence on to the plane wave solution, we were led to consider T-dualities in null or timelike directions. In this chapter I pick up this question, but in a more general context. The spacetime we will study has a parameter J (angular momentum) which may serve as a regulator, allowing investigation of T-dualities that may become null or timelike in a controlled way.

I will first introduce the particular supergravity solution, and then consider various T-duals of it. Next, I will investigate probes in the resulting solutions, and finally discuss the issue of closed timelike curves in these spacetimes.

4.1 Introduction

The model I will consider in this chapter is the type IIb supergravity solution corresponding to a rotating D1-D5-pp system [62]. This is a bound system of D1- and D5-branes with pp-wave (parallel plane wave) excitation in one direction, and with an angular momentum (rotation). It is a 10-dimensional uplift of the arbitrary charge generalisation [63] of the 5-dimensional BMPV black hole solution [64]. The bosonic fields of the supergravity solution are [62] (the below form of the RR-fields was given in ref. [65]):

$$\begin{aligned} ds^2 =&(H_1H_5)^{-\frac{1}{2}}\Big[-(1-\frac{r_p^2}{r^2})dt^2+(1+\frac{r_p^2}{r^2})dz^2-\frac{r_p^2}{r^2}2dtdz \\ &+\frac{J\sin^2\theta}{r^2}(dz-dt)d\phi_1-\frac{J\cos^2\theta}{r^2}(dz-dt)d\phi_2\Big] \\ &+(H_1/H_5)^{\frac{1}{2}}V^{\frac{1}{2}}ds^2_{\mathcal{M}}+(H_1H_5)^{\frac{1}{2}}(dr^2+r^2(d\theta^2+\sin^2\theta d\phi_1^2+\cos^2\theta d\phi_2^2)), \end{aligned} \tag{4.1a}$$

$$e^{2\phi} =g_s^2\frac{H_1}{H_5}, \tag{4.1b}$$

$$\begin{aligned} C^{(2)} =&H_1^{-1}dt\wedge dz+\frac{J}{2r^2}H_1^{-1}(\sin^2\theta d\phi_1-\cos^2\theta d\phi_2)\wedge dz, \\ C^{(6)} =&H_5^{-1}dt\wedge dz\wedge\varepsilon_{\mathcal{M}}+\frac{J}{2r^2}H_5^{-1}(\sin^2\theta d\phi_1-\cos^2\theta d\phi_2)\wedge dz\wedge\varepsilon_{\mathcal{M}}, \end{aligned} \tag{4.1c}$$

where the harmonic functions H_1, H_5 and H_p are given by

$$H_1=1+\frac{r_1^2}{r^2}; \qquad H_5=1+\frac{r_5^2}{r^2}; \qquad H_p=1+\frac{r_p^2}{r^2}, \tag{4.2}$$

and $\mathcal{M}$ is a four-dimensional manifold which can be T^4 or $K3$. The asymptotic volume of $\mathcal{M}$ is denoted V, and $ds^2_{\mathcal{M}}$ is the unit volume element on $\mathcal{M}$. The z-direction is a circle of length $2\pi R_z$, and the angular coordinates take values according to $\theta\in[0,\frac{\pi}{2}]$, $\phi_1\in[0,2\pi)$, $\phi_2\in[0,2\pi)$. The parameter J is the angular momentum, while the parameters r_1 and r_5 quantify the D1- and D5-brane charges, and r_p quantifies the pp-wave charge. The antisymmetric B-field vanishes in this solution.

The metric has a horizon at $r=0$, and the part of spacetime within the horizon is not accessible in these coordinates.

In order to check that the solution is consistent with the supergravity equations of motion, it is necessary to Hodge-dualise the 6-form $C^{(6)}$ to get a 2-form, and then add it to the other 2-form. That is, write $d\tilde{C}^{(2)}=dC^{(2)}+*dC^{(6)}$, which gives

$$\tilde{C}^{(2)}=H_1^{-1}dt\wedge dz+r_5^2\cos^2\theta d\phi_1\wedge d\phi_2+H_1^{-1}\frac{J}{2r^3}r(\sin^2\theta d\phi_1-\cos^2\theta d\phi_2)\wedge(dz-dt). \tag{4.3}$$

Because it will ease the notation later on, let me define the three functions T, S

and U by

$$T(r) = H_1 H_5 H_p - \frac{J^2}{4r^6}, \tag{4.4}$$

$$S(r,\theta) = H_1 H_5 H_p - \frac{J^2 \sin^2\theta}{4r^6} \geq T(r), \tag{4.5}$$

$$U(r,\theta) = H_1 H_5 H_p - \frac{J^2 \cos^2\theta}{4r^6} \geq T(r). \tag{4.6}$$

Before proceeding, it is worth making some observation about the properties of this solution. First, consider the Killing vector

$$\vec{l} = \partial_z + \alpha(\partial_{\phi_1} - \partial_{\phi_2}), \qquad \alpha = -\frac{J}{2r^4}(H_1 H_5)^{-1}, \tag{4.7}$$

with a length

$$\begin{aligned} l^2 &= (H_1 H_5)^{-\frac{3}{2}}\Big[H_1 H_5 H_p - \frac{J^2}{4r^6} + (H_1 H_5)^2 r^2 (\alpha + \frac{J}{2r^4}(H_1 H_5)^{-1})^2\Big] \\ &= (H_1 H_5)^{-\frac{3}{2}} T(r), \end{aligned} \tag{4.8}$$

for the chosen function α. (The α was chosen to make the last term in the above expression vanish, *i.e.* to give a "maximally timelike" vector.)

Define

$$J_* \equiv 2 r_1 r_5 r_p. \tag{4.9}$$

If $J > J_*$ we can have $T(r) < 0$, and the vector $\vec{l}$ is a timelike Killing vector for $r < r_{ch}(J)$, where $r_{ch}(J)$ is defined by the equation $T(r_{ch}) = 0$. Since z is a compact direction, the vector $\vec{l}$ can be the tangent of a closed curve. And it being timelike therefore means there are *closed timelike curves* (CTCs). For this reason the radius $r_{ch}(J)$ is referred to as the *chronology horizon.* Solutions with $J < J_*$ are called under-rotating, while those with $J > J_*$ are called over-rotating. I will return to the issue of closed timelike curves in section 4.4.

Another observation concerns the 5-dimensional spacetime resulting from a Kaluza-Klein reduction on $\mathcal{M}$ and on z. This spacetime is the three-charge generalisation of the BMPV black hole. The entropy of this black hole, as usual given by the area of the horizon, is [62]:

$$S_{5D} = \frac{\pi^2}{2G_5}\sqrt{J_*^2 - J^2}, \tag{4.10}$$

where G_5 is the 5-dimensional Newton's constant. The entropy becomes imaginary in the over-rotating case, $J > J_*$.

4.2 T-dualities

As I will come back to again later, the rotating D1-D5-pp solution and its T-duals have compact directions which become timelike if the rotation J is above the critical value J_*. This is interesting, since we can view the parameter J as a regulator, allowing us a window into the physics of T-duality along a null direction. After one T-duality transformation the potentially timelike compact direction is a coordinate isometry direction, making possible a particularly simple approach to the study of T-duality along a null direction.

The idea is therefore to do a second T-duality along this potentially timelike direction, assuming it is spacelike (*i.e.*,. assuming $J < J_*$). Then, at the end, we can investigate the limit where J tends to, or become greater than J_*. Taking this limit on the component fields will lead to divergences, but it is nevertheless possible that appropriate probes will see a non-singular spacetime. Such probe calculations will be carried out in section 4.3.

Now, let us focus on T-duality transformations. A T-duality transformation along some coordinate direction χ gives the dual NSNS fields (denoted with a tilde) [66, 67]:

$$
\begin{aligned}
\tilde{G}_{\chi\chi} &= \frac{1}{G_{\chi\chi}}, &\quad \tilde{G}_{i\chi} &= \frac{B_{i\chi}}{G_{\chi\chi}}, &\quad \tilde{G}_{ij} &= G_{ij} - \frac{G_{i\chi}G_{j\chi} - B_{i\chi}B_{j\chi}}{G_{\chi\chi}}, \\
& & \tilde{B}_{i\chi} &= \frac{G_{i\chi}}{G_{\chi\chi}}, & \tilde{B}_{ij} &= B_{ij} - \frac{B_{i\chi}G_{j\chi} - G_{i\chi}B_{j\chi}}{G_{\chi\chi}}, \\
e^{2\tilde{\Phi}} &= \frac{e^{2\Phi}}{G_{\chi\chi}},
\end{aligned}
\tag{4.11}
$$

where the index i takes values for all directions except χ. The dual RR potentials are [68]:

$$
\begin{aligned}
\tilde{C}^{(n)}_{i\dots jk\chi} &= C^{(n-1)}_{i\dots jk} - (n-1)\frac{C^{(n-1)}_{[i\dots j|\chi}G_{|k]\chi}}{G_{\chi\chi}} \\
\tilde{C}^{(n)}_{i\dots jkl} &= C^{(n+1)}_{i\dots jkl\chi} + nC^{(n-1)}_{[i\dots jk}B_{l]\chi} + n(n-1)\frac{1}{G_{\chi\chi}}C^{(n-1)}_{[i\dots j|\chi}B_{|k|\chi}G_{|l]\chi}.
\end{aligned}
\tag{4.12}
$$

With these rules at hand, the computations of the T-dual solutions are straightforward.

4.2.1 T-dual along z direction

The T-duality transformation along z (which I will denote T_z) has been discussed in ref. [69], and the resulting solution of type IIa supergravity is:

$$\begin{aligned} ds^2 =&(H_1H_5)^{-\frac{1}{2}}H_p^{-1}\Big[-dt^2+2r\frac{J}{2r^3}(-\sin^2\theta d\phi_1+\cos^2\theta d\phi_2)dt\\ &2r^2\frac{J^2\cos^2\theta\sin^2\theta}{4r^6}d\phi_1 d\phi_2+S(r,\theta)r^2\sin^2\theta d\phi_1^2+U(r,\theta)r^2\cos^2\theta d\phi_2^2\Big]\\ &+(H_1/H_5)^{\frac{1}{2}}V^{\frac{1}{2}}ds^2_{\mathcal{M}}+(H_1H_5)^{\frac{1}{2}}\Big[H_p^{-1}dz^2+dr^2+r^2d\theta^2\Big], \end{aligned} \tag{4.13a}$$

$$B =H_p^{-1}\Big[-(H_p-1)dt+\frac{J}{2r^2}(\sin^2\theta\ d\phi_1-\cos^2\theta\ d\phi_2)\Big]\wedge dz, \tag{4.13b}$$

$$e^{2\Phi} =g_s^2H_1^{\frac{3}{2}}H_5^{-\frac{1}{2}}H_p^{-1}. \tag{4.13c}$$

$$\begin{aligned} C^{(1)} =&H_1^{-1}\Big[dt+\frac{J}{2r^2}(\sin^2\theta\ d\phi_1-\cos^2\theta\ d\phi_2)\Big],\\ C^{(3)} =&H_1^{-1}\frac{J}{2r^2}(\sin^2\theta\ d\phi_1-\cos^2\theta\ d\phi_2)\wedge dt\wedge dz,\\ C^{(5)} =&H_5^{-1}\Big[dt+\frac{J}{2r^2}(\sin^2\theta\ d\phi_1-\cos^2\theta\ d\phi_2)\Big]\wedge\epsilon_{\mathcal{M}}\\ C^{(7)} =&H_5^{-1}\frac{J}{2r^2}(\sin^2\theta\ d\phi_1-\cos^2\theta\ d\phi_2)\wedge dt\wedge\epsilon_{\mathcal{M}}\wedge dz. \end{aligned} \tag{4.13d}$$

This solution has the same horizon at $r=0$ as before the T-duality.

Notice that if $S(r,\theta)<0$ then the ϕ_1 direction becomes timelike. This happens for small r if $J>J_*$. Similarly, the ϕ_2 direction becomes timelike if $U<0$, which is possible also if $J>J_*$.

4.2.2 T-dual along ϕ_1 direction

Since there is no dependence on ϕ_1 in the original solution (4.1a), we can also T-dualise it in the ϕ_1-direction (which is everywhere spacelike) using the rules (4.11, 4.12). To get the right dimensions, define $d\chi\equiv R_{\phi_1}d\phi_1$, where R_{ϕ_1} is some constant.

The resulting type IIa supergravity solution is

$$
\begin{aligned}
ds^2 =& -(H_1H_5)^{-\frac{3}{2}}\Big(H_1H_5(1-\frac{r_p^2}{r^2})+\frac{J^2\sin^2\theta}{4r^6}\Big)dt^2 \\
&+(H_1H_5)^{-\frac{3}{2}}\Big(H_1H_5(1+\frac{r_p^2}{r^2})-\frac{J^2\sin^2\theta}{4r^6}\Big)dz^2 \\
&-(H_1H_5)^{-\frac{3}{2}}\Big(H_1H_5\frac{r_p^2}{r^2}-\frac{J^2\sin^2\theta}{4r^6}\Big)2dtdz+(H_1H_5)^{-\frac{1}{2}}\frac{R_{\phi_1}^2}{r^2\sin^2\theta}d\chi^2 \\
&-\frac{J\cos^2\theta}{2r^3}r(H_1H_5)^{-\frac{1}{2}}2(dz-dt)d\phi_2+(\frac{H_1}{H_5})^{\frac{1}{2}}V^{\frac{1}{2}}ds^2_{\mathcal{M}} \\
&+(H_1H_5)^{\frac{1}{2}}(dr^2+r^2(d\theta^2+\cos^2\theta d\phi_2^2)),
\end{aligned}
\tag{4.14a}
$$

$$
B=(H_1H_5)^{-1}\frac{J}{2r^3}\frac{R_{\phi_1}}{r}(dz-dt)\wedge d\chi,
\tag{4.14b}
$$

$$
e^{2\Phi}=g_s^2H_5^{-\frac{3}{2}}H_1^{\frac{1}{2}}\frac{R_{\phi_1}^2}{r^2\sin^2\theta}.
\tag{4.14c}
$$

$$
\begin{aligned}
C^{(1)} =& -\frac{J}{2r^3}\frac{r}{R_{\phi_1}}H_1^{-1}\sin^2\theta dz, \\
C^{(3)} =& H_1^{-2}H_5^{-1}(H_1H_5+\frac{J^2\sin^2\theta}{4r^6})dt\wedge dz\wedge d\chi+\frac{J}{2r^2}H_1^{-1}\cos^2\theta dz\wedge d\phi_2\wedge d\chi, \\
C^{(5)} =& -\frac{J}{2r^3}\frac{r}{R_{\phi_1}}H_5^{-1}\sin^2\theta dz\wedge\varepsilon_{\mathcal{M}}, \\
C^{(7)} =& H_1^{-1}H_5^{-2}(H_1H_5+\frac{J^2\sin^2\theta}{4r^6})dt\wedge dz\wedge\varepsilon_M\wedge d\chi \\
&+\frac{J}{2r^3}H_5^{-1}r\cos^2\theta dz\wedge\varepsilon_{\mathcal{M}}\wedge d\phi_2\wedge d\chi.
\end{aligned}
\tag{4.14d}
$$

The horizon at $r=0$ is still present. We can write $G_{zz}=(H_1H_5)^{-\frac{3}{2}}S$. So now the z direction is timelike whenever $S(r,\theta)<0$. Again, for this to happen we must have $J>J_*$. Because of the symmetries, a T-duality along ϕ_2 essentially gives the same as a T-duality along ϕ_1, except the condition for the z direction being timelike becomes $U(r,\theta)<0$.

4.2.3 T-dual along both ϕ_1 and z directions

As already mentioned, it is interesting to perform a second T-duality, starting from either of the above solutions of type IIa supergravity. I will consider starting from the T_{ϕ_1}-dual solution, and then take the T-dual along z, giving what I shall call the $T_{\phi_1 z}$-dual solution. As pointed out above, the direction z in the T_{ϕ_1}-dual solution may be null or timelike if $J\geq J_*$, in which case the T-duality transformation $\mathrm{T}_{\phi_1 z}$ is not really well defined. However, we assume first that $J<J_*$, and can then do

the T-dualities without encountering any conceptual problems as the z direction is then everywhere spacelike. Finally, we may investigate what happens as $J \longrightarrow J_*$ or $J > J_*$. Hopefully, this will provide some insight into the problematic issue of T-duality along null or timelike directions.

Performing the T-duality transformation gives the type IIb supergravity solution:

$$\begin{aligned}
ds^2 = &-(H_1H_5)^{\frac{1}{2}}S^{-1}dt^2 + (H_1H_5)^{\frac{3}{2}}S^{-1}dz^2 - (H_1H_5)^{\frac{1}{2}}S^{-1}\frac{J}{2r^3}\frac{R_{\phi_1}}{r}2d\chi dz \\
&+ (H_1H_5)^{\frac{1}{2}}S^{-1}T(r)r^2\cos^2\theta d\phi_2^2 \\
&+ (H_1H_5)^{\frac{1}{2}}H_pS^{-1}\frac{R_{\phi_1}^2}{r^2\sin^2\theta}d\chi^2 + (H_1H_5)^{\frac{1}{2}}S^{-1}\frac{J\cos^2\theta}{2r^2}2dtd\phi_2 \\
&+ H_1^{\frac{1}{2}}H_5^{-\frac{1}{2}}V^{\frac{1}{2}}ds_{\mathcal{M}}^2 + (H_1H_5)^{\frac{1}{2}}(dr^2 + r^2d\theta^2),
\end{aligned} \tag{4.15a}$$

$$e^{2\Phi} = H_1^2S^{-1}\frac{R_{\phi_1}^2}{r^2\sin^2\theta}, \tag{4.15b}$$

$$\begin{aligned}
B = &-S^{-1}(S - H_1H_5)dt\wedge dz - S^{-1}(H_1H_5)\frac{J\cos^2\theta}{2r^3}rd\phi_2\wedge dz \\
&- S^{-1}\frac{J}{2r^3}\frac{R_{\phi_1}}{r}dt\wedge d\chi + S^{-1}\frac{J^2\cos^2\theta}{4r^6}R_{\phi_1}d\phi_2\wedge d\chi.
\end{aligned} \tag{4.15c}$$

$$\begin{aligned}
C^{(0)} = &-H_1^{-1}\frac{J\sin^2\theta}{2r^3}\frac{r}{R_{\phi_1}}, \\
C^{(2)} = &-S^{-1}(S - H_1H_5)H_1^{-1}\frac{J\sin^2\theta}{2r^3}\frac{r}{R_{\phi_1}}dt\wedge dz + S^{-1}H_5\frac{J^2\sin^2\theta}{4r^6}\frac{r^2}{R_{\phi_1}}\cos^2\theta dz\wedge d\phi_2 \\
&- S^{-1}H_5H_pdt\wedge d\chi - S^{-1}H_5H_p\frac{J\cos^2\theta}{2r^3}rd\chi\wedge d\phi_2, \\
C^{(4)} = &-S^{-1}H_5H_p\frac{J\cos^2\theta}{2r^3}rdt\wedge dz\wedge d\chi\wedge d\phi_2 - H_5^{-1}\frac{J\sin^2\theta}{2r^3}\frac{r}{R_{\phi_1}}\epsilon_M, \\
C^{(6)} = &-S^{-1}(S - H_1H_5)H_5^{-1}\frac{J\sin^2\theta}{2r^3}\frac{r}{R_{\phi_1}}dt\wedge dz\wedge\epsilon_M \\
&+ S^{-1}H_1\frac{J^2\sin^2\theta}{4r^6}\frac{r^2}{R_{\phi_1}}\cos^2\theta dz\wedge d\phi_2\wedge\epsilon_M, \\
&- S^{-1}H_1H_pdt\wedge d\chi\wedge\epsilon_M - S^{-1}H_1H_p\frac{J\cos^2\theta}{2r^3}rd\chi\wedge d\phi_2\wedge\epsilon_M, \\
C^{(8)} = &-S^{-1}H_1H_p\frac{J\cos^2\theta}{2r^3}rdt\wedge dz\wedge d\chi\wedge d\phi_2\wedge\epsilon_M.
\end{aligned} \tag{4.15d}$$

This spacetime again has the same horizon at $r = 0$ as before. The components have additional divergences when $S(r,\theta) = 0$, reflecting the fact that the second T-duality was taken along a null direction in those cases. Note also that the Ricci curvature scalar diverges as $S \longrightarrow 0$ (which is evident from plots, although the exact analytic expression is complicated and difficult to analyse). Also, the dilaton Φ blows up in this same limit. These facts tell us that the supergravity solution is not to be trusted in this region as stringy corrections are probably important.

Still, it might be that the supergravity contains some hints about what the true spacetime is. This has been the case before, *e.g.* with the enhançon mechanism [70], where a D-brane probe calculation demonstrated how to resolve a naked singularity in a supergravity solution. In the present situation as well, a natural thing to investigate is therefore how string theory probes behave in the above supergravity solution.

4.3 Probe calculations in $\mathrm{T}_{\phi_1 z}$-dual solution

It has proved very useful in the past to study probes like strings and D-branes which are natural objects in string theory to learn important lessons about properties of various supergravity solutions. The question I will address in this section is whether such probes see the apparent divergent behaviour of the $T_{\phi_1 z}$-dual spacetime at $S = 0$. If the probes are well defined independently of the value of $S(r, \theta)$, this might tell us that the spacetime in reality is freed from singularities by talking into account stringy effects represented by the probes. And if so, that might also indicate that the T-dualities in reality are well-defined throughout the space for all values of the parameter J.

We shall find the result to be that the divergences associated with $S = 0$ cancel from all the probe calculations.

4.3.1 Fundamental string coupled to B-field

The action for a fundamental string coupled to the B-field can be written

$$I = \int d^2\sigma \sqrt{-\det \mathcal{P}(g_{ab})} + \int \mathcal{P}(B), \tag{4.16}$$

where $\mathcal{P}(g_{ab})$ is the induced metric, and $\mathcal{P}(B)$ is the induced B-field on the string worldsheet.

Consider a string whose worldsheet extends in the tz-plane, and use static gauge where position in transverse directions depend on t only, $x^m = x^m(t)$. The induced metric is

$$\mathcal{P}(g_{ab}) = g_{ab} + g_{am}\partial_b X^m + g_{mn}\partial_a X^m \partial_b X^n, \tag{4.17}$$

where $a, b = t, z$, and m, n run over all other (*i.e.*, transverse) directions. Straightforward calculations reveal that

$$\begin{aligned}\det\mathcal{P}(g_{ab}) &= \det g_{ab} + 2(g_{zz}g_{tm} - g_{tz}g_{zm})\dot{X}^m + (g_{zz}g_{mn} - g_{mz}g_{nz})\dot{X}^m\dot{X}^n + \mathcal{O}(\dot{X}^3),\\ \det g_{ab} &= -\,(H_1H_5)^2S^{-2}.\end{aligned} \tag{4.18}$$

In the slow motion approximation the square root can be expanded, leading to

$$\begin{aligned}\sqrt{-\det\mathcal{P}(g_{ab})} = &(H_1H_5)S^{-1} - (H_1H_5)S^{-1}\frac{J\cos^2\theta}{2r^2}\dot{\phi}_2\\ &-\frac{1}{2}(H_1H_5)^{\frac{1}{2}}g_{ij}v^iv^j - \frac{1}{2}\frac{R_{\phi_1}^2}{r^2\sin^2\theta}\dot{\chi}^2 - \frac{1}{2}(H_1H_5)r^2\cos^2\theta\dot{\phi}_2^2,\end{aligned} \tag{4.19}$$

where $v^i = \dot{x}^i$, and i,j now run over all transverse directions except r and ϕ_2. The induced B-field has only one component $\mathcal{P}(B)_{tz}$ which becomes

$$\begin{aligned}\mathcal{P}(B)_{tz} &= B_{\mu\nu}\partial_t X^\mu\partial_z X^\nu = B_{tz} + B_{mz}\dot{X}^m\\ &= -1 + (H_1H_5)S^{-1} - (H_1H_5)S^{-1}\frac{J\cos^2\theta}{2r^2}\dot{\phi}_2\end{aligned} \tag{4.20}$$

Putting things together we see that the potential divergences coming from S^{-1} terms cancel:

$$\sqrt{-\det\mathcal{P}(g_{ab})} - \mathcal{P}(B)_{tz} = 1 - \frac{1}{2}(H_1H_5)^{\frac{1}{2}}g_{ij}v^iv^j - \frac{1}{2}\frac{R_{\phi_1}^2}{r^2\sin^2\theta}\dot{\chi}^2 - \frac{1}{2}(H_1H_5)r^2\cos^2\theta\dot{\phi}_2^2 \tag{4.21}$$

The divergence in the induced metric $\mathcal{P}(g_{ab})$ cancels against the divergence in the induced antisymmetric field B. So this action is everywhere finite (in the slow motion approximation) regardless of J.

A similar calculation for a string in the $t\chi$-plane gives

$$\begin{aligned}\sqrt{-\det\mathcal{P}(g)} + \mathcal{P}(B)_{t\chi} = &\frac{R_{\phi_1}}{r\sin\theta}\Big((H_1H_2H_p)^{\frac{1}{2}} + \frac{J\sin\theta}{2r^3}\Big)^{-1}\Big(1 - \frac{J\cos^2\theta}{2r^3}r\dot{\phi}_2\Big)\\ &-\frac{1}{2}\frac{R_{\phi_1}}{r\sin\theta}(H_1H_5H_p)^{\frac{1}{2}}\Big[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\cos^2\theta\dot{\phi}_2^2\\ &+ H_p^{-1}\dot{z}^2 + H_5^{-1}V^{\frac{1}{2}}v_{\mathcal{M}}^2\Big],\end{aligned} \tag{4.22}$$

which again is finite independently of $S(r,\theta)$.

So the divergences in the supergravity field components seem to be invisible to the strings. This is analogous to the result of ref. [71], where the authors considered the Penrose-Güven scaling limit of the $AdS_5 \times S^5/\mathbb{Z}_N$ spacetime (where N is an integer that tends to infinity in the limit). A T-dualty was performed along a spacelike direction that is periodic due to the $\mathbb{Z}_2$ identification, and which becomes null in the Penrose-Güven limit. The resulting spacetime was found to contain divergent components in the metric and B-field, while string propagation turned out to be finite, due to cancellations between the metric and the B-field parts. Moreover, it was shown to give a non-relativitic string propagating in a background with Newtonian potential. Except for it giving a non-relativistic string, which is due to the scaling limit rather than the null T-duality, this is in complete agreement with the calculations I have done above. This is evidence that T-duality in null directions is not inherently problematic, but only becomes so when we try to apply the supergravity T-duality rules. In the next two subsections I will support this argument by finding the same conclusion by considering two different types of D-brane probes in the $T_{\phi_1 z}$-dual background.

4.3.2 D1-D5-brane probe

Consider a probe consisting of a bound D1-D5-brane state, and view it as an effective string which is fixed on $\mathcal{M}$. (See ref. [65] where such a probe was considered in the original solution (4.1a).) This is a natural choice, knowing that D1- and D5-branes are the basic constituents of this spacetime. The T-dualities have of course complicated things, as is apparent from all the RR-potentials in eq. (4.15d), but let us nevertheless consider this choice of probe. The action is

$$\begin{aligned} I = &\int_\Sigma d^2\xi e^{-\Phi}(k_5 V(r) + k_1)\sqrt{-\det \mathcal{P}(g_{ab} + B_{ab})} \\ &+ k_5 \int_{\Sigma\times M}\Big(C^{(6)} + C^{(4)}\wedge B + \frac{1}{2!}C^{(2)}\wedge B\wedge B + \frac{1}{3!}C^{(0)}B\wedge B\wedge B\wedge B\Big) \\ &+ k_1\int_\Sigma \Big(C^{(2)} + C^{(0)}B\Big), \end{aligned} \tag{4.23}$$

where k_1 and k_2 are constants. (In the $\mathcal{M}$=K3 case they are $k_1 = \tau_1(n_1 - n_5)$ and $k_5 = \tau_5 n_5$, where n_1 and n_5 are the numbers of D1-branes and D5-branes respectively.)

Integrating over the manifold $\mathcal{M}$, and using the fact that $V(r) = \int_{\mathcal{M}}\sqrt{\det g} = H_1 H_5^{-1} V$, this can be written as

$$I = \int d^2\xi (k_1 H_1 + k_5 H_5 V)\Big(-\frac{1}{2}H_p v_1^2 - \frac{1}{2}\dot{z}^2\Big), \tag{4.24}$$

where $v_1^2 = \dot{r}^2 + r^2(\dot{\theta}^2 + \cos^2\theta\dot{\phi}_2^2)$.

This result is very similar to what the same D1-D5-brane probe calculation [65] gives for the original solution. The S^{-1} divergences cancel in the Dirac-Born-Infeld (DBI) part and the Wess-Zumino (WZ) part independently, and the effective action is again perfectly finite throughout spacetime.

4.3.3 D3-brane probe

Next, let us consider a D3-brane probe in this spacetime, and choose a static gauge in which the brane is aligned in the t, z, χ, ϕ_2 directions. The transverse directions are $x^m = x^m(t)$. The action for a general D3-brane probe is

$$\begin{aligned} I &= I_{DBI} + I_{WZ} \\ &= \int_\Sigma d^4\xi e^{-\Phi}\sqrt{-\det \mathcal{P}(g_{ab} + B_{ab})} + \int_\Sigma \mathcal{P}\Big(C^{(4)} + C^{(2)} \wedge B + \frac{1}{2!}C^{(0)} B \wedge B\Big). \end{aligned} \tag{4.25}$$

The various contributions to the Wess-Zumino part are:

$$\begin{aligned} \mathcal{P}(C^{(4)}) &= -dt \wedge dz \wedge d\chi \wedge d\phi_2 \; S^{-1} H_5 H_p \frac{J\cos^2\theta}{2r^3} r \\ \mathcal{P}(C^{(2)} \wedge B) &= +dt \wedge dz \wedge d\chi \wedge d\phi_2 \; S^{-1} H_1^{-1} \frac{J\cos^2\theta}{2r^3} r (H_1 H_5 H_p + \frac{J^2 \sin^2\theta}{4r^6}) \\ \mathcal{P}(\frac{1}{2}C^{(0)} B \wedge B) &= -dt \wedge dz \wedge d\chi \wedge d\phi_2 \; S^{-1} H_1^{-1} \frac{J\cos^2\theta}{2r^3} \frac{J^2 \sin^2\theta}{4r^6} r \end{aligned} \tag{4.26}$$

The different terms cancel against each other so that the WZ part of the action is in fact zero, $I_{WZ} = 0$.

Now, consider the DBI part. The determinant is found to be

$$\det \mathcal{P}(g_{ab} + B_{ab}) = S^{-1}\frac{\cos^2\theta}{\sin^2\theta} R_{\phi_1}^2 \Big[\underbrace{(H_1 H_5 H_p - 2H_1 H_5 - \frac{J^2\cos^2\theta}{4r^6})}_{\equiv -Z} + (H_1 H_5)^{\frac{3}{2}} v^2 \Big], \tag{4.27}$$

where $v^2 = g_{mn}\dot{x}^m \dot{x}^n$ and m, n run over directions transverse to the D3-brane. In the slow motion approximation this gives

$$I_{DBI} = \int d^4\xi e^{-\Phi} S^{-\frac{1}{2}} \frac{\cos\theta}{\sin\theta} R_{\phi_1} \sqrt{Z - (H_1 H_5)^{\frac{3}{2}} v^2}. \tag{4.28}$$

As long as Z is everywhere positive the square root can be expanded, giving

$$S_{DBI} = \int d^4\xi \Big(r\cos\theta H_1^{-1} Z^{\frac{1}{2}} - \frac{1}{2} r\cos\theta Z^{-\frac{1}{2}} H_1^{\frac{1}{2}} H_5^{\frac{3}{2}} v^2 + \mathcal{O}(v^3) \Big), \tag{4.29}$$

which is everywhere finite. This result resembles the previous probe calculations, showing that there is nothing exotic happening with the probe action as the T-duality becomes null ($S = 0$) or even timelike ($S < 0$).

However, the function Z is not necessarily everywhere positive. It is given as

$$Z(r) = H_1 H_5 (2 - H_p) + \frac{J^2 \cos^2\theta}{4r^6} = H_1 H_5 \frac{r^2 - r_p^2}{r^2} + \frac{J^2 \cos^2\theta}{4r^6}, \tag{4.30}$$

which can be negative or zero if $r^2 < r_p^2$. This behaviour of the D3-brane probe does not depend on J, and has nothing to do with the second T-duality being null. In fact, as can easily be seen from the expression above, Z becomes negative most easily if $J = 0$.

If the function Z is negative, the above action is imaginary. To see what that implies, consider a static D3-brane ($v^2 = 0$). Then, zero Z means vanishing action, *i.e.* the effective D3-brane tension vanishes, meaning that the D3-brane probe action as written above is not valid anymore. The reason for this happening is that the B-field starts dominating over the metric field g, so that the determinant $\det \mathcal{P}(g + B)$ becomes positive. The physical interpretation is that the D-brane becomes effectively tensionless, and in fact tachyonic for $\det \mathcal{P}(g + B) > 0$ [72].

4.4 Closed timelike curves

I mentioned in section 4.1 that the rotating D1-D5-pp spacetime has closed timelike curves (CTCs). These are "time-loops" which lead to obvious paradoxes of the sort that you could go back in time and kill your own grand-father. Some general discussions on CTCs are found in refs. [73–75], while recent work related to the BMPV solution include refs. [76–78].

This section contains a brief discussion of CTCs in the various solutions described in section 4.2. They will appear in all solutions in the over-rotating case, although the T-duality transformations change which directions correspond to CTCs. I will also address the question whether any of these curves may be geodesics.

Original solution We saw in section 4.1 that the original solution has a Killing vector which is spacelike, null or timelike depending on the sign of $T(r)$. And since it is tangent to a closed curve, there are closed timelike curves (CTCs) if $T(r) < 0$.

This is possible if $J > J_*$ within the chronology horizon, $r < r_{ch}(J)$. These closed timelike curves are trivial in the sense that they are resolved by going to the universal covering space (where z is non-compact, so that although the curve is timelike, it is never closed) [62]. However, the CTCs are there also after T-duality transformations, and cannot generically be resolved by going to the universal covering space.

$\mathbf{T}_z$-dual solution In this solution there are potential closed timelike curves given by the tangent vectors

$$\vec{l}_1 = \partial_{\phi_1}, \qquad \vec{l}_2 = \partial_{\phi_2}. \tag{4.31}$$

In the following I will discuss $\vec{l}_1$ only, as the treatment of $\vec{l}_2$ is completely analogous. The length of $\vec{l}_1$ is

$$l_1^2 = (H_1 H_5)^{-\frac{1}{2}} H_p^{-1} r^2 \sin^2\theta \ S(r,\theta), \tag{4.32}$$

which is timelike if $S(r,\theta) < 0$ ($\vec{l}_2$ is timelike if $U < 0$). For this to happen, we still need $J > J_*$, the same requirement as before T-duality. However, since S depends on θ, the chronology horizon $r'_{ch}(\theta) \leq r_{ch}$ will in this case depend on θ: For $\theta \to 0$, $r'_{ch} \to 0$, while for $\theta \to \pi/2$, $r'_{ch} \to r_{ch}$ This CTC is not resolved by going to the universal covering space.

$\mathbf{T}_{\phi_1}$-dual solution In this solution there is an obvious closed timelike curve (for large enough J) given by the tangent vector

$$\vec{l} = \partial z, \qquad l^2 = (H_1 H_5)^{-\frac{3}{2}} S(r,\theta), \tag{4.33}$$

which again is timelike if $S(r,\theta) < 0$.

$\mathbf{T}_{\phi_1 z}$-dual solution If $J < J_*$ then all directions but time are everywhere spacelike, and no CTCs may appear. So let us assume that $J > J_*$. Then this spacetime can be divided into three regions. The inner one where $S < 0$, $T < 0$; a middle one where $T < 0 < S$; and an outer one where $S > 0$, $T > 0$. If $S \leq 0$, there are divergences/flips of signs, corresponding to the last T-duality being null/timelike. Let us for the moment not dwell on this, but simply assume that $S > 0$. For these parts of spacetime at least, the T-duality should not be problematic, since the directions along which we have T-dualised are always spacelike. In the outer region, where also $T > 0$, there are no timelike directions except time. Hence, I will consider the middle region where $S(r,\theta) > 0$ but $T(r) < 0$. The tangent vector

$$\vec{l} = \partial_{\phi_2}, \qquad l^2 = r^2 \cos^2\theta (H_1 H_5)^{-\frac{1}{2}} \frac{T}{S} \tag{4.34}$$

is then timelike. And since ϕ_2 is a compact direction, this means there are CTCs. This is not very surprising, as it is just the previous CTCs showing up in another place.

In summary it appears that T-duality does little to change the chronology structure, merely shifting the closed timelike curves around. This is an intuitive result, as a T-duality transformation is exactly what the name says: a duality. So the physics before and after should be the same. This result is in line with refs. [79–81] where CTCs are found in both the Gödel solution and its T-dual which is a *compactified* plane wave solution.

4.4.1 CTCs and geodesics

It is not completely clear that the mere existence of closed timelike curves necessarily is a pathology of a supergravity solution. The arguments saying that CTCs represent unphysical situations, are usually based on considerations of matter travelling along the CTCs. But it is perceivable that there can be CTCs along which no matter can travel, and thus some of the problems with defining field theory in spaces with CTCs are avoided. However, if a CTC is a geodesic, then matter can certainly travel along it, and the problems seem unavoidable. It is therefore an interesting question to ask whether the above found CTCs can be geodesics.

Such questions have been asked in refs. [79, 80] in the context of Gödel-like universes. and it was found that none of the CTCs were geodesics. However, there is also evidence [82] that probe geodesics (the trajectory of a supertube) in the Gödel-like universe may in fact close.

In the following I will do a similar analysis to the one done in ref. [80] for the solutions at hand, and demonstrate that the same result emerges: None of the closed timelike curves are geodesics.

For a curve to be a geodesic, its tangent vector $\vec{l}$ needs to satisfy the geodesic equations:

$$G^a \equiv \dot{l}^a - \Gamma^a_{bc} l^b l^c = 0. \tag{4.35}$$

Original solution The non-trivial geodesic equations for $\vec{l} = \partial_z + \alpha(\partial_{\phi_1} - \partial_{\phi_2})$ are

$$\begin{aligned} G^r &= \frac{1}{2r}(H_1 H_5)^{-3}\Big[(H_1 + H_5)(H_1 H_5 H_p - \frac{3J^2}{4r^6}) - 2(H_1 H_5)^2\Big], \\ G^{\phi_1} &= \dot{\alpha}, \qquad G^{\phi_2} = -\dot{\alpha}. \end{aligned} \tag{4.36}$$

The last two equations, $\dot{\alpha} = \frac{d\alpha}{dr}\dot{r} = 0$, require $\dot{r} = 0$, so only circular motion may be geodesic, and the requirement of the Killing vector being a geodesic becomes

$$F(r) \equiv (H_1 + H_5)(H_1 H_5 H_p - \frac{3J^2}{4r^6}) - 2(H_1 H_5)^2 = 0 \tag{4.37}$$

for some constant r. The important feature of this function is that generically, when $F(r) = 0$ we have $T(r) > 0$. In other words, all geodesics of this family are spacelike.

We could also try to study geodesics in more generality by considering the effective Lagrangian of a point-particle in the spacetime along the lines of [80]. In the present case, however, there is less symmetry to simplify the calculations, and this approach leads to complicated differential equations which I have not been able to solve.

T_z-dual solution The non-trivial geodesic equations for the vector $\vec{l_1} = \partial_{\phi_1}$ are

$$\begin{aligned} G_1^r &= -\frac{r}{2}(H_1 H_5 H_p)^{-2} \sin^2\theta \Big[(H_1 + H_5) H_1 H_5 H_p^2 \\ &\quad + (H_p(H_1 + H_5) + 2H_1 H_5)\frac{J^2 \sin^2\theta}{4r^6})\Big], \\ G_1^\theta &= -(H_1 H_5 H_p)^{-1} \sin\theta\cos\theta \Big[H_1 H_5 H_p - \frac{2J^2 \sin^2\theta}{4r^6}\Big]. \end{aligned} \tag{4.38}$$

The first equation requires $\theta = 0$, which also agrees with the second equation. However, when this is the case $S(r, 0) = H_1 H_5 H_p > 0$, so the geodesic is spacelike.

T_{ϕ_1}-dual solution The non-trivial geodesic equations for the vector $\vec{l} = \partial_z$ are

$$\begin{aligned} G^r &= \frac{1}{2r}(H_1 H_5)^{-3}\Big[(H_1 + H_5)(H_1 H_5 H_p - \frac{3J^2}{4r^6}) - 2(H_1 H_5)^2\Big], \\ G^\theta &= \frac{1}{r^2}(H_1 H_5)^{-2}\frac{J^2 \sin^2\theta \cos^2\theta}{4r^6}. \end{aligned} \tag{4.39}$$

The second equation requires $\theta = 0$ or $\theta = \frac{\pi}{2}$, and the first equation gives the same constant r as in the original solution, again giving a closed curve which is spacelike rather than timelike or null. So in conclusion, none of these CTCs can be geodesics.

$\mathbf{T}_{\phi_1 z}$-dual solution The non-trivial geodesic equations for the tangent vector $\vec{l} = \partial_{\phi_2}$ are

$$\begin{aligned} G^r &= -\frac{r\cos^2\theta}{2S^2 H_1 H_5}\Big[(H_1+H_5)T^2 + \frac{J^2\cos^2\theta}{4r^6}\Big((H_1+H_5)(2H_1H_5H_p+T)+2(H_1H_5)^2\Big)\Big], \\ G^{\phi_2} &= \frac{T^2}{S^2}\sin\theta\cos\theta. \end{aligned} \tag{4.40}$$

The second equation is satisfied if (i) $T=0$, (ii) $\theta=\frac{\pi}{2}$ or (iii) $\theta=0$. Combined with the first equation, this gives for case (i) that we have to require $\theta=\frac{\pi}{2}$, which gives a geodesic solution without any restriction on r. However, for this value of θ we have $S=T$, and since we assumed $S>0$ we only have $T>0$. That is, we only get spacelike geodesics. The same holds for case (ii). Case (iii) leads to the equation

$$\frac{J^2}{4r^6}\Big(H_1H_5H_p(H_1+H_5)+2(H_1H_5)^2\Big)+(H_1H_5H_p)^2(H_1+H_5)=0, \tag{4.41}$$

which is never satisfied (the left hand side is always positive). So, again, we find no CTCs which are geodesics.

An interesting question to ask is whether the CTCs not being geodesics is a T-duality invariant statement. These calculations suggest so, and also those of ref. [80]. But these examples are of course not enough to state this as a general fact, and it would be very interesting to study this further.

4.5 Discussion

Timelike T-dualities have been studied in refs. [83, 84], and was shown to relate type IIa (IIb) string theories to so-called type IIb* (IIa*) string theories. The type II* supergravities are similar to type II, except for sign differences. These are nonetheless important differences, and imply that type II* supergravities have ghosts. In the gravity/gauge duality picture, they correspond to non-unitary gauge theories also with ghosts. However, these ghosts are believed to be an artefact of the supergravity truncation, and the full type II* string theories are supposed to be ghost-free.

We have seen that the the existence of closed timelike curves is not affected by the T-dualities, which supports intuition based on the fact that a T-duality transformation does not change the physics. As long as we stay in the under-rotating regime $J < J_*$ the T-dual solutions are not very exotic – they share the same properties as the original solutions, but are just more complicated. If we approach the over-rotating regime $J > J_*$, we have seen that there are divergences appearing in the components of the $T_{\phi_1 z}$-dual solution that are associated with a T-duality transformation being taken along a null direction. This is an expected result, as it follows immediately from the transformation rules (4.11). A more interesting result is the fact that various probes put into this spacetime are free of the same divergence. A possible interpretation of this is that the divergence is an artefact of the supergravity truncation that is somehow resolved by properly taking into account α' corrections incorporated in the probes. Essentially, we ought to remember that the elementary objects in the geometry are not points as assumed in the supergravity approximation, but extended strings and branes. Furthermore, the result suggests that T-duality transformations in null or even timelike directions make sense with the same corrections taken into account.

We have also seen that none of the CTCs satisfy the geodesic equation, and are therefore not geodesics. However, this does not necessarily mean that no CTCs can be geodesics, since there might be other CTCs in these spacetimes than those I have discussed. A general investigation of the geodesic by a Lagrangian approach would make the answer definite. But in these spacetimes, with so few symmetries, such a study is highly non-trivial.

The problem of closed timelike curves is one of the prime motivations for the work of the remaining part of this thesis. In chapter 6 I will describe an exact model where it is possible to compute all α' corrections to the supergravity solution, and see how those change the discussion of CTCs. Before going into details on that, however, a more general discussion of Wess-Zumino-Witten-Novikov models is necessary.

Chapter 5

Wess-Zumino-Novikov-Witten models

In the previous chapter we saw the existence of closed timelike curves in a solution to supergravity, but did not dwell on the significance or interpretation of such curves. One immediate question that comes to mind is how full string theory might modify this picture: Will the problems associated with closed timelike curves be resolved in string theory beyond the supergravity limit?

For the remaining part of this thesis I will study an exact string theory model (and a generalisation of it) where this question can be addressed properly and to some extent answered by direct calculations. But before turning to the particular models, I will in this chapter present the necessary concepts and techniques that are the basis for the next two chapters. First, I will introduce the Wess-Zumino-Novikov-Witten model in section 5.1, and how it is related to coset constructions in two-dimensional conformal field theory in section 5.2. Section 5.3 is a presentation of a method for extracting the exact spacetime fields from coset models. Then in section 5.4 I discuss heterotic coset models, which are the type of constructions we will use in the subsequent chapters. Section 5.5 gives an example of how the ideas are applied in the case of a two-dimensional charged black hole.

5.1 WZNW models

Wess-Zumino-Novikov-Witten (WZNW) models [85–87] are conformal field theories (CFTs) in which a current algebra gives the spectrum of the theory. They can be given a Lagrangian formulation as nonlinear sigma models on a group manifold with an antisymmetric tensor background, and provide a very useful setting in which to study many aspects of string theory, as is apparent from the large amount of attention such models have attracted. In string theory, these models correspond to bosonic strings propagating on a group manifold. An introduction to WZNW models can be found in ref. [88].

Such models were first studied in the context of two-dimensional bosonisation, giving the bosonisation rules for non-interacting massless fermions [85]. It was demonstrated that a theory with N massless free Majorana fermions is equivalent to a theory with N massless free bosons with global symmetry $O(N)$, which can be expressed as a WZNW model based on the group $O(N)$.

Current algebras Consider a general CFT with a set of $(1,0)$ currents $j^a(z)$. If we expand the currents as a Laurent series,

$$j^a(z) = \sum_m j^a_m z^{-m-1}, \tag{5.1}$$

then the conformal invariance implies that the modes satisfy the following algebra

$$[j^a_m, j^b_n] = i f^{ab}{}_c j^c_{m+n} + mkd^{ab}\delta_{m+n}. \tag{5.2}$$

Note that $[j^a_0, j^b_0] = i f^{ab}{}_c j^c_0$, so $f^{ab}{}_c$ are the structure constants of the Lie algebra satisfied by the zero modes. The term proportional to k is a *central extension*, often referred to as a *Schwinger term*. The factor d^{ab} is the Lie algebra inner derivative, defined as $d^{ab} = (T^a, T^b)$, with $\{T^a\}$ being a basis for the algebra. The constant k is the *level* constant.

The algebra (5.2) is known as a *current algebra*, or *affine Lie algebra*, or *affine Kac-Moody algebra*. It is common to denote this algebra $\hat{G}_k$, where $\hat{G}_0 = \mathrm{Lie}(G)$ is the associated Lie algebra corresponding to a Lie group G.

We can study the current algebra and find out lots of things about the theory. Particularly useful is the so-called *Sugawara construction* which gives the energy-momentum tensor as a product of two currents. Essentially, it says that the Virasoro algebra belongs to the enveloping algebra of the current algebra. The exact connection between the energy-momentum tensor and the currents is a remarkable feature of current algebras with conformal symmetry in two dimensions.

I will not go into any more detail on this, but refer to refs. [14,88,89] for thorough treatments of current algebras, and other aspects of conformal field theory.

Although much can be found out by studying the current algebra, it is also very useful to have a Lagrangian formulation of the theory. And this is exactly what the WZNW models provide. Consider a Lie group G, and let $g \in G$ be a map from the worldsheet Σ to the group G, $g : \Sigma \to G$. Then the WZNW action is

$$S = kI_{WZNW}(g) = -\frac{k}{4\pi}\int_\Sigma d^2z \mathrm{Tr}(g^{-1}\partial g g^{-1}\bar{\partial}g) + ik\Gamma(g), \tag{5.3}$$

where

$$\Gamma(g) = \frac{1}{12\pi}\int_{\mathcal{B}} \mathrm{Tr}(g^{-1}dg)^3 \tag{5.4}$$

is known as a *Wess-Zumino* term. The constant k is the same as the level constant in the current algebra (5.2). I have used the notation $\partial = \partial_z$, $\bar{\partial} = \partial_{\bar{z}}$. Σ is the embedding of the worldsheet in the group manifold G, and $\mathcal{B}$ is a three-dimensional surface whose boundary is Σ. The Wess-Zumino term is a three-dimensional integral, but since the integrand can be written as a total derivative it can *locally* (albeit not globally) be rewritten as an integral over the worldsheet Σ.

To see explicitly that the WZNW action can be written as a standard nonlinear sigma model, we should parameterise the group element as $g = e^{iT_\mu X^\mu}$, where T_μ are generators of the group. The WZNW action then becomes

$$\begin{aligned} I &= \frac{k}{4\pi}\int_\Sigma d^2z \mathrm{Tr}(T_\mu T_\nu)\partial X^\mu \bar{\partial} X^\nu + \frac{1}{12\pi}\int_{\mathcal{B}} \mathrm{Tr}(T_\mu T_\nu T_\rho) dX^\mu \wedge dX^\nu \wedge dX^\rho \\ &= \frac{k}{4\pi}\int_\Sigma d^2z \mathrm{Tr}(T_\mu T_\nu)\partial X^\mu \bar{\partial} X^\nu + \frac{1}{12\pi}\int_{\mathcal{B}} d^3\sigma \mathrm{Tr}(T_\mu T_\nu T_\rho)\epsilon^{abc}\partial_a X^\mu \partial_b X^\nu \partial_c X^\rho \\ &= \frac{k}{4\pi}\int_\Sigma d^2z \Big(\mathrm{Tr}(T_\mu T_\nu) + \mathrm{Tr}(T_{[\mu}T_{\nu]}T_\rho)\epsilon^{z\bar{z}}X^\rho\Big)\partial X^\mu \bar{\partial} X^\nu. \end{aligned} \tag{5.5}$$

This is of the form of the nonlinear sigma model

$$S = \frac{1}{2\pi\alpha'}\int d^2z (G_{\mu\nu} + B_{\mu\nu})\partial X^\mu \bar{\partial} X^\nu, \tag{5.6}$$

with the metric $G_{\mu\nu} = \mathrm{Tr}(T_\mu T_\nu)$, and antisymmetric B-field $B_{\mu\nu} = \mathrm{Tr}(T_{[\mu}T_{\nu]}T_\rho)X^\rho$. The level constant k should be identified with the string tension, $k \sim \frac{1}{\alpha'}$. So the low-energy regime $\alpha' \to 0$ corresponds to $k \to \infty$.

Despite this connection, there is one property of the WZNW models not shared in general by nonlinear sigma models that makes them particularly interesting. Namely the fact that they are described by an exact CFT. Nonlinear sigma models are string theories formulated in a particular background, and valid to first order in α'. On the other hand, WZNW models, through the CFT description, are well defined independently of any background spacetime. It has a purely worldsheet formulation valid for any value of α'.

5.2 Coset models as gauged WZNW models

Coset models were invented [90, 91] in the early seventies, and later generalised [92, 93] as algebraic realisations of new conformal systems G/H based upon current algebras for a group G and a subgroup H.

An important property of CFTs with current algebra is that they can always be factored into two completely independent parts [14]: One Sugaware part and one part that commutes with the current algebra. That is, the CFT splits into a Sugawara theory, with an energy-momentum tensor T^s obtained via the Sugawara construction entirely from the current, and another CFT with an energy-momentum tensor T' that commutes with the current. The total energy-momentum tensor can then be written $T = T^s + T'$.

In the case of a current algebra $\hat{G}$ and a subalgebra $\hat{H} \subset \hat{G}$, this means that we can split up the energy-momentum tensor $T^{\hat{G}} = T^{\hat{H}} + T^{\hat{G}/\hat{H}}$. For any subalgebra, the Sugawara theory separates into the Sugawara theory of the subalgebra, and a new *coset CFT*.

A representation χ_r of the $\hat{G}$ current algebra can be decomposed under the subalgebra,

$$\chi_r^{\hat{G}} = \sum_{r',r''} n^r_{r'r''} \chi^{\hat{H}}_{r'} \chi^{\hat{G}/\hat{H}}_{r''}, \tag{5.7}$$

where r' and r'' runs over all representations of $\hat{H}$ and $\hat{G}/\hat{H}$ respectively, and $n^r_{r'r''}$ are non-zero integers. Since current algebra theories are quite well understood, this is often a very useful way to represent the coset theory.

Within the Lagrangian formulation, the coset construction can be regarded as a gauging of the subalgebra H. Conformal invariance forbids a kinetic term for the gauge field, and the equations of motion for this field then requires the H-charge to vanish, leaving only the coset theory.

The coupling to the gauge fields A is straightforward for the metric part of the WZNW action. We simply use minimal coupling, and replace derivatives with covariant derivatives, roughly $\partial \to D = \partial + A$. The metric part is then gauge invariant by itself.

The Wess-Zumino term $\Gamma(g)$, on the other hand, poses some difficulty. It does not allow a gauge invariant extension for general subgroup H. However, there *is* a unique extension which is such that its variation under a gauge transformation only depends on the gauge fields and not on the group element $g \in G$ [94].

Assume that the ungauged model has the symmetry group $G_L \times G_R$, where the subscripts L and R denote left and right action respectively. This means that the action is invariant under the global transformation

$$g(z,\bar{z}) \to \tilde{g}_L\, g(z,\bar{z})\, \tilde{g}_R^{-1}, \qquad \tilde{g}_L, \tilde{g}_R \in G. \tag{5.8}$$

We want to promote this to a local symmetry. In other words, we want the action to be invariant under local transformation in a subgroup $H \subset G$, *i.e.*, we impose the symmetry

$$g(z,\bar{z}) \to h_L(z,\bar{z})\, g(z,\bar{z})\, h_R^{-1}(z,\bar{z}), \qquad h_L, h_R \in H. \tag{5.9}$$

In the special case that $h_R^{-1} = h_L$, this is called an *axial gauging*, and if $h_R = h_L$ it is called a *vector gauging*. To realise this symmetry, introduce the gauge fields A_a^i and the covariant derivative

$$D_a g = \partial_a g + A_{L,a} g - g A_{R,a} = \partial_a g + A_a^i (T_L^i g - g T_R^i), \tag{5.10}$$

where T_L^i, T_R^i ($i = 1, \ldots, \dim(H)$) are generators of the subgroup H, and I am using the notation $A_L = A_a^i T_L^i d\sigma^a = A_{L,a} d\sigma^a$ and $A_R = A_a^i T_R^i d\sigma^a = A_{R,a} d\sigma^a$.

The unique extension [94] of the Wess-Zumino term is then

$$\begin{aligned}\Gamma(g) \to \Gamma(g) + \frac{1}{4\pi} \int \Big[& A_a^i d\sigma^a \wedge \mathrm{Tr}(T_L^i dg g^{-1} + T_R^i g^{-1} dg) \\ & + \frac{1}{2} A_a^i A_b^j d\sigma^a \wedge d\sigma^b \mathrm{Tr}(T_R^i g^{-1} T_L^j g - T_R^j g^{-1} T_L^i g) \Big].\end{aligned} \tag{5.11}$$

Putting this together, the action for the gauged WZNW model [95–100] becomes[1]

$$\begin{aligned}I(g,A) = k I_{WZNW}(g) + \frac{2k}{4\pi} \int d^2z \mathrm{Tr}\Big(& A_{L,\bar{z}} \partial g g^{-1} - A_{R,z} g^{-1} \bar{\partial} g - A_{L,\bar{z}} g A_{R,z} g^{-1} \\ & + \frac{1}{2}(A_{L,z} A_{L,\bar{z}} + A_{R,z} A_{R,\bar{z}}) \Big)\end{aligned} \tag{5.12}$$

Since there is no gauge-invariant extension for the Wess-Zumino term $\Gamma(g)$ for general subgroup H, this action has (in general) classical anomalies. Only certain subgroups and gaugings therefore give sensible theories, singled out by the condition of anomaly cancellation.

Consider the gauge variation $\delta A_a^i = \partial_a \alpha^i$, $\delta g = \alpha^i (T_L^i g - g T_R^i)$. This leads to the

[1] Expressions in the literature often differ from this one by the interchange $z \leftrightarrow \bar{z}$, which amounts to subtracting rather than adding the extension to $\Gamma(g)$, or to define $\epsilon^{z\bar{z}} = +1$ rather than -1.

anomaly

$$\delta I(g, A) = \alpha \frac{k}{2\pi} \mathrm{Tr}(T_L^i T_L^i - T_R^i T_R^i) \int d^2z F_{z\bar{z}}. \tag{5.13}$$

So the anomaly cancellation condition is

$$\mathrm{Tr}(T_L^i T_L^i - T_R^i T_R^i) = 0. \tag{5.14}$$

By using gauged WZNW models, actions can be written for a large class of conformal field theories obtained as coset models.

Let me briefly recapitulate: The ungauged model has some global symmetry group G which defines a conformal field theory [101–103] with an underlying current algebra. Coupling it to gauge fields charged under a subgroup $H \subset G$ gives the coset, as reviewed above. Such models have been used to generate conformal field theories for many studies in string theory, including cosmological contexts. It is important to note that the vast majority of these models use a particular sort of gauging. The basic worldsheet field $g(z, \bar{z})$ is group valued, and the full global invariance is $G_L \times G_R$, realised as: $g(z, \bar{z}) \to g_L g(z, \bar{z}) g_R^{-1}$, for $g_L, g_R \in G$. The sorts of group actions gauged in most studies are $g \to h_L g h_R^{-1}$, for $h_L, h_R \in H$, and it is only a restricted set of choices of the action of h_L and h_R which allows for the writing of a gauge invariant action. These are the "anomaly-free" subgroups, and the typical choice that is made is to correlate the left and right actions so that the choice is essentially left-right symmetric (axial or vector gauging). This also gives a symmetric structure on the worldsheet, as appropriate to bosonic strings and to superstrings if one considers supersymmetric WZNW models. For these anomaly-free subgroups, a gauge extension of the basic WZNW action can be written which is H-invariant, and the resulting conformal field theory is well-defined. The supersymmetric models can of course be turned into heterotic string theories too, by simply tensoring with the remaining conformal field theory structures needed to make a left-right asymmetric model. I will return to heterotic models in section 5.4.

5.3 Exact spacetime fields from coset models

In this section I will review the procedure for extracting the exact spacetime fields from the gauged WZNW model.

The idea [104] is very simple: Integrate out the gauge fields, compare the resulting action to the nonlinear sigma model, and read off the fields. However, if we do this directly, the result is only going to be valid to first order in the parameter k, since the procedure of integrating out the gauge fields in the naive way is only valid to first order. The reason for this is that we are treating the gauge fields as classical fields, substituting their on-shell behaviour into the action to derive the effective nonlinear sigma model action for the rest of the fields, and ignoring the effects of quantum fluctuations arising at subleading order in the large k expansion.

To include all of the physics and derive a result valid at any order in k, we need to do better than this. This sort of thing has been achieved before, using a number of methods. The first time in ref. [105] in the context of the $SL(2,\mathbb{R})/U(1)$ coset model studied as a model of a two-dimensional black hole [104]. (See also section 5.3.4.) The exact metric and dilaton were deduced by appealing to a group theoretic argument, writing the exact expressions for the quadratic Casimirs for G and for H, in terms of the target space (G/H) fields, and then equating their difference to the Laplacian for the propagation of a massless field (the tachyon) in the background. The proposed metric and dilaton were verified at higher orders by explicit calculation in refs. [106,107], and the argument was generalised and applied to a number of other models in a series of papers [108,109].

An elegant alternative method was developed in refs. [110,111], and is the one I will adapt for use in the following. The basic idea of this method is to *first* deduce a quantum effective action which takes into account the higher energy modifications of the classical action, and *then* integrate out the gauge fields. Since the effective action already encodes high energy effects, this integration can be done accurately by simply solving the equations of motion and substituting back. Done this way, the fields we read off will be valid to all orders in k.

5.3.1 The Exact Effective Action

The first thing to do is therefore to write down the effective action, which in general is not a simple task. But it is known [110,112,113] that the exact effective action for the ungauged WZNW model defined in eq. (5.3) is extremely simple to write down: Take the form of the basic action at level k, $kI(g)$, where $g \in G$ is a quantum field, and write for the full quantum effective action $(k - c_G)I(g)$, where now g should be taken as a classical field, and c_G is the dual Coxeter number of the group G. This is particularly simple since k only enters the action as an overall multiplicative factor, which then gets shifted, *i.e.*,

$$kI_{WZNW}(g) \to (k - c_G)I_{WZNW}(g). \tag{5.15}$$

The key observation of refs. [110,111] is that this can be applied to a gauged WZNW model as well, by exploiting the fact that if we write $A_z = \partial_z h_z h_z^{-1}$ and

$A_{\bar{z}} = \partial_{\bar{z}} h_{\bar{z}} h_{\bar{z}}^{-1}$, the action can be written as the sum of two formally decoupled WZNW models, one for the field $g' = h_{\bar{z}}^{-1} g h_z$ at level k and the other for the field $h' = h_{\bar{z}}^{-1} h_z$ at level $2c_H - k$. To write the exact effective action, shift the levels in each action: $k \to k - c_G$ and $2c_H - k \to 2c_H - k - c_H = c_H - k$, and treat the fields as classical. Transforming back to the original variables, we get the original gauged WZNW model with its level shifted according to $k \to k - c_G$, together with a set of new terms for $A_z, A_{\bar{z}}$ which are proportional to $c_H - c_G$, and have no k dependence. Because there is no multiplicative factor of k in these new terms, it is easy to see that the large k contribution to the result of integrating out the gauge fields will be the same as as if we had used the classical action. For results exact in k, there will be a family of new contributions to the nonlinear sigma model couplings upon integrating out the gauge fields. In this effective action, they are to be treated as classical fields now and so once the integration is done, there are no further contributions from quantum fluctuations to take into account. The metrics derived using this method are the same as those constructed using the algebraic approach, which is a useful consistency check [110, 111].

Note that the new pieces in the effective action are non-local in the fields $A_z, A_{\bar{z}}$ (although local in the $h_z, h_{\bar{z}}$). This difficulty does not present a problem for the purposes of reading off the spacetime fields, since it is enough to work in the zero-mode sector of the string to capture this information. This amounts to dropping all derivatives with respect to σ on the world-sheet and working with the reduced "point–particle" Lagrangian for that aspect of the computation [111].

5.3.2 Computation of the Exact Effective Action

Let us assume that the complete classical action is a $G/H = G_1 \times \cdots \times G_N/H$ gauged WZNW action. A group element $g \in G$ can then be split up

$$g = \begin{pmatrix} g_1 & 0 & \cdots & 0 \\ 0 & g_2 & & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & & g_N \end{pmatrix}, \tag{5.16}$$

and the action becomes a sum of WZNW actions

$$S = \sum_m k_{(m)} I_{WZNW}(g_m), \tag{5.17}$$

with $g_m \in G_m$.

We are going to gauge a subgroup $H \subset G$, and so we introduce the covariant derivative as defined in eq.(5.10). The connections A_a^i are the gauge fields, which take values in the Lie algebra of H. The $T_{i,L}$ are left generators, and $T_{i,R}$ are right generators of H. Using the block diagonal notation above, we can write

$$A = A^i \begin{pmatrix} T_i^{(1)} & & 0 \\ & \ddots & \\ 0 & & T_i^{(N)} \end{pmatrix} \in \mathrm{Lie}(H). \tag{5.18}$$

The gauged WZNW model (5.12) is then also a sum of terms corresponding to each factor G_m in the group G,

$$S_{gWZNW} = \sum_m k_{(m)} \left[I_{WZNW}(g_m) + S_1(g_m, A) \right] \tag{5.19}$$

where

$$\begin{aligned} S_1(g, A) = \frac{2}{4\pi} \int d^2z \mathrm{Tr} \Big\{ & A_{\bar{z},L} \partial_z g g^{-1} - A_{z,R} g^{-1} \partial_{\bar{z}} g - A_{\bar{z},L} g A_{z,R} g^{-1} \\ & + \frac{1}{2} (A_{z,L} A_{\bar{z},L} + A_{z,R} A_{\bar{z},R}) \Big\} . \end{aligned} \tag{5.20}$$

The anomaly cancellation condition (5.14) generalises to

$$\sum_{m,i} k_{(m)} \mathrm{Tr}(T_{i,L}^{(m)} T_{i,L}^{(m)} - T_{i,R}^{(m)} T_{i,R}^{(m)}) = 0. \tag{5.21}$$

A Change of variables

By the change of variables

$$\begin{aligned} A_{z,L} &= -\partial_z h_z h_z^{-1} , \quad A_{z,R} = -\partial_z \tilde{h}_z \tilde{h}_z^{-1} , \qquad h, \tilde{h} \in H , \\ A_{\bar{z},L} &= -\partial_{\bar{z}} h_{\bar{z}} h_{\bar{z}}^{-1} , \quad A_{\bar{z},R} = -\partial_{\bar{z}} \tilde{h}_{\bar{z}} \tilde{h}_{\bar{z}}^{-1} , \end{aligned} \tag{5.22}$$

we find

$$\begin{aligned} S_1(g, h) = \frac{2}{4\pi} \int d^2z \mathrm{Tr} \Big\{ & - \partial_z g g^{-1} \partial_{\bar{z}} h_{\bar{z}} h_{\bar{z}}^{-1} + g^{-1} \partial_{\bar{z}} g \partial_z \tilde{h}_z \tilde{h}_z^{-1} - \partial_{\bar{z}} h_{\bar{z}} h_{\bar{z}}^{-1} g \partial_z \tilde{h}_z \tilde{h}_z^{-1} g^{-1} \\ & + \frac{1}{2} (\partial_{\bar{z}} h_{\bar{z}} h_{\bar{z}}^{-1} \partial_z h_z h_z^{-1} + \partial_{\bar{z}} \tilde{h}_{\bar{z}} \tilde{h}_{\bar{z}}^{-1} \partial_z \tilde{h}_z \tilde{h}_z^{-1}) \Big\} . \end{aligned} \tag{5.23}$$

The Polyakov-Wiegmann identity [86, 114] implies:

$$
\begin{aligned}
I(h_{\bar z}^{-1} g \tilde h_z) =& I(g) + I(h_{\bar z}^{-1}) + I(\tilde h_z) \\
&+ \frac{2}{4\pi} \int d^2z \mathrm{Tr}\Big[-\partial_{\bar z} h_{\bar z} h_{\bar z}^{-1} \partial_z g g^{-1} - \partial_{\bar z} h_{\bar z} h_{\bar z}^{-1} g \partial_z \tilde h_z \tilde h_z^{-1} g^{-1} + g^{-1} \partial_{\bar z} g \partial_z \tilde h_z \tilde h_z^{-1} \Big] , \\
I(h_{\bar z}^{-1} h_z) =& I(h_{\bar z}^{-1}) + I(h_z) + \frac{2}{4\pi} \int d^2z \mathrm{Tr}\Big[-\partial_{\bar z} h_{\bar z} h_{\bar z}^{-1} \partial_z h_z h_z^{-1} \Big] , \\
I(\tilde h_{\bar z}^{-1} \tilde h_z) =& I(\tilde h_{\bar z}^{-1}) + I(\tilde h_z) + \frac{2}{4\pi} \int d^2z \mathrm{Tr}\Big[-\partial_{\bar z} \tilde h_{\bar z} \tilde h_{\bar z}^{-1} \partial_z \tilde h_z \tilde h_z^{-1} \Big] . \qquad (5.24)
\end{aligned}
$$

Using these, the classical action (5.19) can be written as

$$
\begin{aligned}
S_1 =& - I(g) + I(h_{\bar z}^{-1} g \tilde h_z) - \frac{1}{2}\Big[I(h_{\bar z}^{-1} h_z) + I(\tilde h_{\bar z}^{-1} \tilde h_z) \Big] - \frac{1}{2} C \Big] , \\
&\text{where} \quad C \equiv I(h_{\bar z}^{-1}) - I(\tilde h_{\bar z}^{-1}) - I(h_z) + I(\tilde h_z) .
\end{aligned}
$$

The term C is not manifestly gauge invariant, but the others are. Note that if $A_L = A_R$, then $C = 0$, in which case the gauging is classically anomaly-free. Otherwise, the anomalous terms C_i may look disturbing, but in fact they cancel in the action, $\sum k_{(i)} C_i = 0$, as follows from the condition of anomaly cancellation (5.21).

The change of variables (5.22) gives rise to a Jacobian which can be re-written as an additional term in the action, proportional to the dual Coxeter number c_H of the subgroup[2],

$$
J = \det(\frac{\delta A}{\delta h}) = \exp\Big\{ 2c_H \frac{1}{2} \Big[I_{WZNW}(h_{\bar z}^{-1} h_z) + I_{WZNW}(\tilde h_{\bar z}^{-1} \tilde h_z) \Big] \Big\} \qquad (5.25)
$$

Taking all this into account, and assuming we have chosen a gauging which is overall non-anomalous, we can write the action as:

$$
S = - \sum_i \Big\{ k_{(i)} I(h_{\bar z}^{-1} g_i \tilde h_z) - (k_{(i)} - 2c_H) \frac{1}{2} \Big[I(h_{\bar z}^{-1} h_z) + I(\tilde h_{\bar z}^{-1} \tilde h_z) \Big] \Big\} . \qquad (5.26)
$$

Note that $h_{\bar z}^{-1} g h_z \in G$, $h_{\bar z}^{-1} h_z \in H$, and $\tilde h_{\bar z}^{-1} \tilde h_z \in H$. As promised in the previous section we have achieved to rewrite of the full action in the form of a sum of ungauged WZNW actions, which allows us to write down the quantum effective action in a very simple way.

[2] I will later only consider cases where the dual Coxeter number is zero, in which cases this Jacobian vanishes.

Effective action

Using the simple prescription given above,

$$
\begin{aligned}
&\text{for } G\text{: } k_{(i)} \rightarrow k_{(i)} - c_{G_i} \ , \\
&\text{for } H\text{: } \ -k_{(i)} + 2c_H \rightarrow (-k_{(i)} + 2c_H) - c_H = -(k_{(i)} - c_H) \ ,
\end{aligned}
\tag{5.27}
$$

we find the effective action

$$
S^{eff} = -\sum_i \Big\{ (k_{(i)} - c_{G_i}) I(h_{\bar{z}}^{-1} g_i \tilde{h}_z) - (k_{(i)} - c_H) \frac{1}{2} \Big[I(h_{\bar{z}}^{-1} h_z) + I(\tilde{h}_{\bar{z}}^{-1} \tilde{h}_z) \Big] \Big\} \ .
\tag{5.28}
$$

Again, the action is manifestly gauge invariant.

Return to the original variables

We now change variables back to the original ones, using the identities given above. The result is

$$
\begin{aligned}
S^{\text{eff}} = -\sum_i \Big\{ &(k_{(i)} - c_{G_i}) \Big[I(g) + S_1(g, A) + \frac{1}{2} [I_2(A_L) + I_2(A_R)] + \frac{1}{2} C_i \Big] \\
&- (k_{(i)} - c_H) \frac{1}{2} \Big[I_2(A_L) + I_2(A_R) \Big] \Big\} \ ,
\end{aligned}
\tag{5.29}
$$

where $I_2(A_L) \equiv I(h_{\bar{z}}^{-1} h_z)$, $I_2(A_R) \equiv I(\tilde{h}_{\bar{z}}^{-1} \tilde{h}_z)$. Observe that the C_i's have come back into the action (in a combination that does not cancel). Rewritten, this is

$$
S^{\text{eff}} = -\sum_i (k_{(i)} - c_{G_i}) \Big[I(g) + S_1(g, A) - \frac{\lambda_i}{2} [I_2(A_L) + I_2(A_R)] \Big] \ ,
\tag{5.30}
$$

where $\lambda_i = \frac{c_{G_i} - c_H}{k_{(i)} - c_{G_i}}$.

Since we consider the fields in the effective action to be classical, there is *no* Jacobian associated with this change of variables.

5.3.3 Extracting the Exact Geometry

As stated earlier, a problem with working with this action is that it has terms which are non-local in the gauge fields. Since we are going to integrate these out, this is inconvenient. To avoid this complication, we reduce to the zero mode sector [111], which is enough to extract the information we want. The zero mode sector is obtained by letting fields depend on worldsheet time only. So ∂_z and $\partial_{\bar{z}}$ are both replaced by ∂_τ. We also denote A by a in this limit. This leads to the desired simplifications. An additional simplification is that the WZ part $\Gamma(g)$ of the WZNW action vanishes in this sector.

The resulting action is

$$S_0^{\text{eff}} = -\sum_i \frac{(k_{(i)} - c_{G_i})}{4\pi} \int d\tau \Big\{ \mathrm{Tr}(g^{-1}\partial g g^{-1}\partial g)$$
$$+ 2\mathrm{Tr}\big[a_{\bar{z},L}\partial g g^{-1} - a_{z,R} g^{-1}\partial g - a_{\bar{z},L} g a_{z,R} g^{-1} + \frac{1}{2}(a_{z,L}a_{\bar{z},L} + a_{z,R}a_{\bar{z},R})\big]$$
$$- \lambda_i \frac{1}{2}\mathrm{Tr}\big[(a_{\bar{z},L} - a_{z,L})^2 + (a_{\bar{z},R} - a_{z,R})^2\big]$$
$$+ \frac{1}{2}\mathrm{Tr}\big[a_{z,R}a_{z,R} - a_{\bar{z},R}a_{\bar{z},R} + a_{\bar{z},L}a_{\bar{z},L} - a_{z,L}a_{z,L}\big] \Big\} . \tag{5.31}$$

This is a local action quadratic in a. It is useful to simplify the notation, so let me define

$$\begin{aligned}
L^a = L^a_M \partial X^M &= \sum_i (k_{(i)} - c_{G_i}) \mathrm{Tr}(T_{R,a} g^{-1}\partial g) , \\
-R^a = -R^a_M \partial X^M &= \sum_i (k_{(i)} - c_{G_i}) \mathrm{Tr}(T_{L,a} \partial g g^{-1}) , \\
M_{ab} &= \sum_i (k_{(i)} - c_{G_i}) \mathrm{Tr}(T_{L,a} g T_{R,b} g^{-1} - T_{L,a}T_{L,b}) , \\
\widetilde{M}_{ab} &= \sum_i (k_{(i)} - c_{G_i}) \mathrm{Tr}(T_{L,b} g T_{R,a} g^{-1} - T_{R,a}T_{R,b}) = M_{ba} + 2H_{ab} , \\
G_{ab} &= \sum_i (k_{(i)} - c_{G_i}) \lambda_i \frac{1}{2} \mathrm{Tr}(T_{L,a}T_{L,b} + T_{R,a}T_{R,b}) \\
&= \sum_i (c_{G_i} - c_H) \frac{1}{2} \mathrm{Tr}(T_{L,a}T_{L,b} + T_{R,a}T_{R,b}) , \\
H_{ab} &= \sum_i (k_{(i)} - c_{G_i}) \frac{1}{2} \mathrm{Tr}(T_{L,a}T_{L,b} - T_{R,a}T_{R,b}) , \\
\mathrm{g} = \mathrm{g}_{MN} \partial X^M \partial X^N &= \sum (k_{(i)} - c_{G_i}) \mathrm{Tr}(g^{-1}\partial g g^{-1}\partial g) .
\end{aligned} \tag{5.32}$$

In this notation the action can be written as:

$$\begin{aligned}
S_0^{\text{eff}} = -\frac{1}{4\pi} \int d\tau \Big\{ &\mathrm{g} - 2a^a_{\bar{z}} R_a - 2a^a_z L_a - 2a^a_{\bar{z}} a^b_z (M_{ab} - G_{ab} + H_{ab}) \\
&- a^a_z a^b_z (G_{ab} + H_{ab}) - a^a_{\bar{z}} a^b_{\bar{z}} (G_{ab} - H_{ab}) \Big\} .
\end{aligned} \tag{5.33}$$

Defining

$$z^i = \begin{pmatrix} a^a_{\bar{z}} \\ a^b_z \end{pmatrix} , \qquad B_i = \begin{pmatrix} R_a \\ L_b \end{pmatrix}^T , \qquad A_{ij} = \begin{pmatrix} G_- & M - G_- \\ \widetilde{M} - G_+ & G_+ \end{pmatrix} , \tag{5.34}$$

where $G_+ = G + H$ and $G_- = G - H$, the action can be written in the compact form

$$S_0^{\text{eff}} = -\frac{1}{4\pi} \int d\tau \Big\{ \mathrm{g} - 2B_i z^i - z^i A_{ij} z^j \Big\} . \tag{5.35}$$

Now we complete the square, and get

$$S_0^{\text{eff}} = -\frac{1}{4\pi} \int d\tau \Big\{ \mathrm{g} - A_{ij}(z + A^{-1}B)^i (z + A^{-1}B)^j + A^{kl} B_k B_l \Big\} , \tag{5.36}$$

where $A^{kl} \equiv (A^{-1})_{kl}$.

The equations of motion for z (*i.e.*, the equations of motion for the gauge fields a_z and $a_{\bar{z}}$) are easily read off,

$$\delta z \Rightarrow \qquad z^i = -A^{ik} B_k . \tag{5.37}$$

Inserting this into the action, we arrive at

$$S_{\text{min}}^{\text{eff}} = -\frac{1}{4\pi} \int d\tau \Big[\mathrm{g} + B_k A^{kl} B_l \Big] . \tag{5.38}$$

To write out this explicitly, we need to invert the matrix A_{ij}. If we write this inverted matrix as

$$A^{-1} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} , \tag{5.39}$$

then equation (5.37) gives

$$a^a_{\bar{z}} = -p_{ab} R_b - q_{ab} L_b , \tag{5.40}$$

$$a^a_z = -r_{ab} R_b - s_{ab} L_b , \tag{5.41}$$

and

$$\begin{aligned} S_{\text{min}}^{\text{eff}} &= -\frac{1}{4\pi} \int d\tau \Big[\mathrm{g} + R^a p_{ab} R^b + R^a (q_{ab} + r_{ba}) L^b + L^a s_{ab} L^b \Big] \\ &= -\frac{1}{4\pi} \int d\tau \Big[\mathrm{g}_{MN} + R^a_M p_{ab} R^b_N + R^a_M (q_{ab} + r_{ba}) L^b_N + L^a_M s_{ab} L^b_N \Big] \partial X^M \partial X^N \\ &= -\frac{1}{4\pi} \int d\tau \ \frac{1}{2} C_{MN} \partial X^M \partial X^N . \end{aligned} \tag{5.42}$$

This action is of the form of the nonlinear sigma-model, and we can extract the exact spacetime fields from the coefficients C_{MN}.

Furthermore, finding the coefficients C_{MN} means finding the matrices p, q, r, s. Explicitly,

$$C_{MN} = 2[g_{MN} + b_{MN} + R^a_M p_{ab} R^b_N + R^a_M (q_{ab} + r_{ba}) L^b_N + L^a_M s_{ab} L^b_N] \ , \tag{5.43}$$

where I have included the antisymmetric field b_{MN} which is the counterpart to g_{MN}, and comes from the WZ term. Its contribution vanishes here in the zero mode sector, but it will be relevant for reading off the B-field. Note that with this definition, C_{MN} is not automatically symmetric.

The metric is now easily read off by symmetrising C_{MN},

$$G_{MN} = C_{(MN)}. \tag{5.44}$$

The dilaton is a one-loop effect which is given by a determinant coming from the integration of the gauge fields,

$$e^{2\hat{\Phi}} = (\det A)^{-\frac{1}{2}}. \tag{5.45}$$

To get the B-field really requires going beyond the zero-mode sector. But there is a trick to do this without any extra work [111]. Simply take advantage of the fact that the local part of the action is

$$S = -\frac{1}{4\pi}\int d^2z \frac{1}{2} C_{MN} \partial X^M \bar{\partial} X^N, \tag{5.46}$$

so that the B-field is given by antisymmetrising C_{MN},

$$B_{MN} = C_{[MN]}. \tag{5.47}$$

Thus we have found general expressions for the exact (in α') spacetime fields G_{MN}, B_{MN} and $\hat{\Phi}$ for coset models.

5.3.4 Example: A 2D black hole

As a simple example of how this works, let us now consider the two-dimensional black hole spacetime [104] corresponding to the coset $SL(2,\mathbb{R})/U(1)$. The exact metric and dilaton were first worked out in ref. [105] using a Hamiltonian approach. The computation has been done over again in ref. [110] using roughly the same method as presented here.

Start by parameterising the group element g, and the gauge symmetry generators T_L, T_R according to

$$g = e^{\frac{t_L}{2}\sigma_3} e^{\frac{r}{2}\sigma_1} e^{\frac{t_R}{2}\sigma_3} = \begin{pmatrix} e^{\frac{t_+}{2}} \cosh\frac{\sigma}{2} & e^{\frac{t_-}{2}} \sinh\frac{\sigma}{2} \\ e^{-\frac{t_-}{2}} \sinh\frac{\sigma}{2} & e^{-\frac{t_+}{2}} \cosh\frac{\sigma}{2} \end{pmatrix} \in SL(2,\mathbb{R}), \tag{5.48}$$

$$T_L = -T_R \equiv T = \frac{1}{2}\sigma_3 \quad \in \mathrm{Lie}(U(1)), \tag{5.49}$$

where $t_\pm = t_L \pm t_R$. This gauging is an axial gauging, $g \to e^T g e^T$, and is classically non-anomalous, which means the anomaly cancellation condition is automatically satisfied. For this coset we have $c_G = 2$, $c_H = 0$, $\lambda = \frac{2}{k-2}$. Let us also choose unitary gauge where $t_L = 0$ (giving $t_+ = t_R$, $t_- = -t_R$).

We then find

$$H = 0, \quad G = (k-2)\lambda \mathrm{Tr}(TT) = (k-2)\lambda\frac{1}{2}, \tag{5.50}$$

and furthermore

$$\begin{aligned} g_{\mu\nu}\partial X^\mu \partial X^\nu &= (k-2)\frac{1}{2}(d\sigma^2 + dt_R^2), \\ M = \tilde{M} &= (k-2)\mathrm{Tr}(TgTg^{-1} - TT) = (k-2)(\cosh^2\frac{\sigma}{2} - 1), \\ L_\mu \partial X^\mu &= (k-2)\mathrm{Tr}(Tg^{-1}\partial g) = (k-2)(\frac{1}{2}\partial t_R), \\ R_\mu \partial X^\mu &= (k-2)\mathrm{Tr}(T\partial g g^{-1}) = (k-2)(\cosh^2\frac{\sigma}{2} - \frac{1}{2})\partial t_R, \end{aligned} \tag{5.51}$$

and

$$\begin{aligned} p = s &= -\frac{G}{M(M-2G)} = \frac{\lambda}{(k-2)2\sinh^2\frac{\sigma}{2}(-\cosh^2\frac{\sigma}{2} + 1 + \lambda)}, \\ q = r &= \frac{M-G}{M(M-2G)} = \frac{-2\cosh^2\frac{\sigma}{2} + 2 + \lambda}{(k-2)2\sinh^2\frac{\sigma}{2}(-\cosh^2\frac{\sigma}{2} + 1 + \lambda)}. \end{aligned} \tag{5.52}$$

This gives the exact metric

$$ds^2 = \frac{k-2}{2}\Big[-\Big(\frac{\cosh\sigma + 1}{\cosh\sigma - 1} - \frac{2}{k}\Big)^{-1} dt_R^2 + d\sigma^2\Big] \tag{5.53}$$

together with a vanishing B-field, and the dilaton

$$e^{-4\hat{\Phi}} = \det A = -\frac{1}{4}(k-2)^2(\cosh\sigma + 1)(\cosh\sigma + 1 + \frac{2}{k-2}). \tag{5.54}$$

5.4 Heterotic coset models

The heterotic coset model technique which I will describe in this section, was introduced in ref. [115] as a way to study more general coset models. It goes beyond the coset construction as described in section 5.2, and exploits the basic fact that the heterotic string is asymmetric in how it is built. The idea is to allow ourselves the freedom to choose to gauge far more general subgroups. This might well produce anomalies, but permits us to choose to keep certain global symmetries which might be of interest (such as spacetime rotations) and/or use in the conformal field theory. Introducing right-moving fermions to achieve a right-moving supersymmetry is easy to do, and they contribute extra terms to the anomaly, making matters worse in general: Their couplings (the effective charges they carry under H) are completely determined by supersymmetry, so we have no choice. Of course, we do not have a well-defined model if there are anomalies, so ultimately they must be eliminated. This is achieved as follows [115]. Note that the left-moving fermions can be introduced with *arbitrary* couplings (charges under H), since there is no requirement of left-moving supersymmetry in the heterotic string. The anomaly they contribute comes with the opposite sign to that of the others, since they have the opposite chirality. The requirement that the anomaly cancels can be satisfied, since it just gives a set of algebraic equations to solve for the charges. The resulting model is a conformal field theory with (0,1) worldsheet supersymmetry, (enhanced to $(0,2)$ when G/H is Kähler [116–118]) naturally adapted to the heterotic string.

It is important to note that the types of heterotic models obtained by this method are very different from the types of models obtained by gaugings that do not cancel the anomalies against those of the gauge fermions. One way to see the difference is to note that since the anomaly is proportional to k, the cancellation equation puts the gauge charge at the same order as the metric. This means that there is a non-trivial modification of the geometry that we would read off from the WZNW action, traceable to the left-moving fermions. We will see this in detail in the following.

The method described in the previous section for extracting the exact spacetime fields in coset models has to be extended to work for heterotic coset models. Although heterotic backgrounds have been considered in some previous works [108], they are of the mildly heterotic type which are essentially similar to the superstring models: an asymmetric arrangement of fermions is merely tensored in as dressing.

The crucial fact that makes the same ideas work also with fermions included, is that the fermions, through non-Abelian bosonisation, can also be written as a gauged WZNW model. The issue of non-Abelian bosonisation was in fact what historically led to the WZNW model in the first place [85].

5.4.1 Abelian bosonisation

The technique of bosonisation is based on the remarkable equivalence in two dimensions between theories of fermions and theories of bosons. The simplest example of this is the equivalence between massless free Dirac fermions and massless free scalar bosons. Before turning to non-Abelian bosonisation, which is really what I am interested in, let me first summarise the basic Abelian bosonisation rules.

In terms of a Dirac spinor Ψ and a real boson Φ the bosonisation rules in two dimensions are

$$\begin{aligned} &\text{kinetic term:} && \bar{\Psi}\gamma^a\partial_a\Psi \leftrightarrow g^{ab}\partial_a\Phi\partial_b\Phi, && (5.55)\\ &\text{vector coupling:} && \bar{\Psi}\gamma^a\Psi \leftrightarrow \epsilon^{ab}\partial_b\Phi, && (5.56)\\ &\text{axial coupling:} && \bar{\Psi}\gamma^a\gamma^5\Psi \leftrightarrow g^{ab}\partial_b\Phi, && (5.57)\end{aligned}$$

where $\bar{\Psi} = \Psi^\dagger\gamma^0$. By using the complex worldsheet coordinates $z = t - ix, \bar{z} = t + ix$, and the basis

$$\gamma^0 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^5 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5.58}$$

we find

$$\gamma^a\partial_a = \begin{pmatrix} 0 & \partial \\ \bar{\partial} & 0 \end{pmatrix}. \tag{5.59}$$

Furthermore, writing

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \tag{5.60}$$

and defining

$$P_\pm = \frac{1}{2}(1 \pm \gamma^5) \qquad \Rightarrow \qquad P_+\Psi = \psi_L, \quad P_+\Psi = \psi_R, \tag{5.61}$$

we get the following kinetic term for the fermions

$$\bar{\Psi}\gamma^a\partial_a\Psi = \psi_L\partial\psi_L + \psi_R\bar{\partial}\psi_R \leftrightarrow \partial\Phi\bar{\partial}\Phi. \tag{5.62}$$

The coupling to a gauge field can then be written

$$\begin{aligned} j^aA_a &= \bar{\Psi}\gamma^a\Psi A_a = \psi_L\psi_L A_z + \psi_R\psi_R A_{\bar{z}} = j_L A_z + j_R A_{\bar{z}}, \\ j_5^a A_a &= \bar{\Psi}\gamma^5\gamma^a\Psi A_a = -\psi_L\psi_L A_z + \psi_R\psi_R A_{\bar{z}} = -j_L A_z + j_R A_{\bar{z}}, \end{aligned} \tag{5.63}$$

where I have identified

$$j_R(z) = \psi_R\psi_R, \qquad j_L(\bar{z}) = \psi_L\psi_L. \tag{5.64}$$

Moreover, this means that with both vector and axial couplings (charges v and a respectively), we can write

$$vj^a A_a + aj_5^a A_a = (v+a)j_R A_{\bar{z}} + (v-a)j_L A_z. \tag{5.65}$$

The bosonisation rules are easily seen from the above to be

$$j_R = \partial\Phi, \qquad j_L = -\bar{\partial}\Phi. \tag{5.66}$$

5.4.2 Non-Abelian bosonisation

The Abelian bosonisation procedure turns out to be very awkward in the case of theories with non-Abelian symmetries. Luckily, there is an alternative, more general approach [85]. First, let us re-write the currents (5.66) so as to make it easier to generalise to the non-Abelian case. Define the field $U = e^{\Phi}$. then we get

$$j_R = U^{-1}\partial U, \qquad j_L = -U^{-1}\bar{\partial}U. \tag{5.67}$$

However, with non-Abelian symmetry this is not correct, since these currents are inconsistent with the conservation laws $\bar{\partial}j_R = 0$, $\partial j_L = 0$. The modification needed is not very big. What we have to worry about in the non-Abelian case is the ordering of the terms in the definition of the currents. If we define a field g which is element in some group G, the proper currents are

$$j_R = g^{-1}\partial g, \qquad j_L = -\bar{\partial}g g^{-1}. \tag{5.68}$$

The question is then what Lagrangian governs this bosonic field g? It turns out that this is exactly the WZNW Lagrangian (5.3) we have discussed earlier. This is the basis of non-Abelian bosonisation – a fermionic free theory with global symmetry $SO(N)$ in two dimensions is equivalent to a WZNW model based on the group $SO(N)$ at level 1 (*i.e.*, with current algebra $\hat{SO}(N)_1$) [85].

For our discussion of heterotic coset models this is very good news. It means that including fermions in our gauged WZNW model is achieved simply by adding an extra WZNW action based on the group $SO(N)$. The complete action for the heterotic coset model can therefore be written as a sum of WZNW actions, thereby allowing us to deduce the exact spacetime fields as described earlier. One important observation in this respect is that the level constant $k = 1$ for the bosonised fermions is *not* shifted when going to the effective action.

5.4.3 Re-fermionisation

The aim of all the calculations described earlier in this chapter was to deduce the exact form of the spacetime fields. For the present heterotic coset model, those are the fields that appear in the *heterotic nonlinear sigma model*:

$$S = \frac{1}{4\pi}\int d^2z\Big[(G_{\mu\nu}+B_{\mu\nu})\partial X^\mu\bar{\partial}X^\nu + G_{\mu\nu}\tilde{\psi}^\mu\mathcal{D}_z\tilde{\psi}^\nu + \lambda^A\mathcal{D}_{\bar{z}}\lambda^A + \frac{1}{2}F^{AB}_{\rho\sigma}\lambda^A\lambda^B\tilde{\psi}^\rho\tilde{\psi}^\sigma\Big] \tag{5.69}$$

where the covariant derivatives are

$$\mathcal{D}_z\tilde{\psi}^\mu = \partial\tilde{\psi}^\mu + (\Gamma^\mu_{\rho\sigma} - \frac{1}{2}H^\mu_{\rho\sigma})\partial X^\rho\tilde{\psi}^\sigma, \tag{5.70}$$

$$\mathcal{D}_{\bar{z}}\lambda^A = \bar{\partial}\lambda^A - iA^{AB}_\mu\bar{\partial}X^\mu\lambda^B. \tag{5.71}$$

The λ's and $\tilde{\psi}$'s are the left- and right-moving fermions respectively, and F and $H^\mu_{\rho\sigma}$ are the field strengths corresponding to the gauge potential A_μ and antisymmetric tensor potential $B_{\mu\nu}$ respectively. The quantity $\Gamma^\mu_{\rho\sigma}$ is the spin connection, associated with the metric $G_{\mu\nu}$.

When reading off the spacetime fields from our heterotic coset model, this is the action we have to compare to. But there is a small problem, since we have the action presented in a fully bosonised form. The idea is therefore to compare our bosonic action (expressed in a suitable form) with a bosonised version of the nonlinear sigma model.

We have seen that the local part of the action can be written (where I have re-introduced dependence on worldsheet space as well as time, which is necessary to deduce the B-field)

$$S = \int d^2z(\mathcal{G}_{MN} + \mathcal{B}_{MN})\partial X^M\bar{\partial}X^N, \tag{5.72}$$

where $\mathcal{G}_{MN} = C_{(MN)}$ and $\mathcal{B}_{MN} = C_{[MN]}$, and the coefficients C_{MN} are given in eq.(5.43).

Assume that the original G/H coset model is a D-dimensional model ($D = \dim(G) - \dim(H)$), and that the fermions bosonise into d bosons. Then this action as it is written is effectively a $(D+d)$-dimensional bosonic theory where the extra d dimensions correspond to the bosonised fermions, Φ_i. But to deduce the correct spacetime fields, we have to take into account that the fermions really are fermions.

This is therefore not the appropriate form of the action for reading off those fields. Instead, we ought to re-write it as a 4-dimensional heterotic nonlinear sigma model, and *then* identify the fields. In order to do that, we need to re-fermionise Φ_i.

For our purpose it is not necessary to perform this re-fermionisation explicitly – it suffices to re-write the action in a form that *prepares* it for re-fermionisation [115]. This is achieved by the following re-writing

$$\begin{aligned} L &= (\mathcal{G}_{MN} + \mathcal{B}_{MN})\partial X^M \bar{\partial} X^N \\ &= (G_{\mu\nu} + B_{\mu\nu})\partial X^\mu \bar{\partial} X^\nu + (\mathcal{G}_{mn} + \mathcal{B}_{mn})\partial\psi^m \bar{\partial}\psi^\nu + \psi^m \mathcal{F}^{(B)}_{m\ z\bar{z}}, \end{aligned} \tag{5.73}$$

where

$$\begin{aligned} \partial\psi^n &= \partial\Phi^n + A^n_\mu \partial X^\mu, \\ A^n_\mu &= \mathcal{G}^{nm} A_{m\mu} = \mathcal{G}^{nm}\mathcal{G}_{m\mu}. \\ B_{\mu n} &= \mathcal{B}_{\mu n} + \mathcal{B}_{mn} A^m_\mu, \\ B_n &= B_{\mu n}\partial X^\mu, \quad \bar{B}_n = B_{\mu n}\bar{\partial} X^\mu, \\ \mathcal{F}^{(B)}_{n\ z\bar{z}} &= \partial \bar{B}_n - \bar{\partial} B_n, \end{aligned} \tag{5.74}$$

This is the wanted form of the action, and the metric and B-field can now be read off. Some simple algebra reveals that

$$\begin{aligned} G_{\mu\nu} &= \mathcal{G}_{\mu\nu} - \mathcal{G}_{mn} A^m_\mu A^n_\nu, \\ B_{\mu\nu} &= \mathcal{B}_{\mu\nu} + (A^m_\mu \mathcal{B}_{\nu m} - A^m_\nu \mathcal{B}_{\mu m}) + \mathcal{B}_{mn} A^m_\mu A^n_\nu. \end{aligned} \tag{5.75}$$

The gauge fields A^n_μ and $B^n_\mu = \mathcal{G}^{nm} B_{\mu m}$ are also given by eqs. (5.74).

One remark is in order at this point: As mentioned above the re-fermionisation transforms the model from a six-dimensional purely bosonic model to a four-dimensional model with both bosons and fermions, which is then to be identified as a heterotic nonlinear sigma model. This of course reminds us of dimensional reduction, and in fact, the analogy is not a coincident [119]. Explicitly, the re-fermionisation corresponds to a reduction with this ansatz for the metric and B-field (see *e.g.* ref. [120]):

$$\mathcal{G}_{MN} = \begin{pmatrix} G_{\mu\nu} + \mathcal{G}_{mn} A^m_\mu A^n_\nu & \mathcal{G}_{mn} A^m_\mu \\ \mathcal{G}_{mn} A^m_\mu & \mathcal{G}_{mn} \end{pmatrix}, \tag{5.76}$$

$$\mathcal{B}_{MN} = \begin{pmatrix} B_{\mu\nu} - 2A^m_{[\mu} B_{\nu]m} + \mathcal{B}_{mn} A^m_\mu A^n_\nu & B_{\mu n} - \mathcal{B}_{mn} A^m_\mu \\ -B_{\nu m} + \mathcal{B}_{mn} A^n_\nu & \mathcal{B}_{mn} \end{pmatrix}. \tag{5.77}$$

The ansatz is found by requiring the model to have the right transformation properties.

The dilaton is generated by the determinant, $\det A$, arising from the integration of the gauge fields (see eq. (5.45)). It also receives a correction due to the re-fermionisation, as is apparent from the above analogy to dimensional reduction. With the decomposition of $\mathcal{G}_{MN}$ given above, we get

$$\sqrt{-\det \mathcal{G}_{MN}} = \sqrt{-\det G_{\mu\nu}}\sqrt{\det \mathcal{G}_{mn}}. \tag{5.78}$$

Now, the dilaton in D dimensions, $\hat{\Phi}$, is related to the dilaton in $D+d$ dimensions, $\hat{\Phi}^{D+d}$, through the equation

$$\sqrt{-\det \mathcal{G}_{MN}}e^{-2\hat{\Phi}^{D+d}} = \sqrt{-\det G_{\mu\nu}}e^{-2\hat{\Phi}}. \tag{5.79}$$

Putting this together, we find that the exact dilaton is

$$e^{2\hat{\Phi}} = (\det A)^{-\frac{1}{2}}(\det \mathcal{G}_{mn})^{-\frac{1}{2}}. \tag{5.80}$$

5.5 A deformation of a charged 2D black hole

To demonstrate how these ideas work, I will now discuss a one-parameter deformation of a charged 2D black hole, which has previously been discussed in the large k limit in ref. [115]. This is also a warm-up before next chapter, since this model is simpler, yet share some features with the 4D model to be discussed there. The heterotic coset model to be considered in this section is

$$G/H = SL(2,\mathbb{R})\times SO(2)/U(1), \tag{5.81}$$

where the $SO(2)$ group is associated with the already bosonised fermions. Let us parameterise $g_b \in SL(2,\mathbb{R})$ by means of Euler angles,

$$g_b = e^{\frac{t_L}{2}\sigma_3}e^{\frac{r}{2}\sigma_1}e^{\frac{t_R}{2}\sigma_3} = \begin{pmatrix} e^{\frac{t_+}{2}}\cosh\frac{\sigma}{2} & e^{\frac{t_-}{2}}\sinh\frac{\sigma}{2} \\ e^{-\frac{t_-}{2}}\sinh\frac{\sigma}{2} & e^{-\frac{t_+}{2}}\cosh\frac{\sigma}{2} \end{pmatrix} \in SL(2,\mathbb{R}). \tag{5.82}$$

Writing $\tilde{g}_L \in H_L$, $\tilde{g}_R \in H_R$, the gauging is chosen as

$$g_b \longrightarrow \tilde{g}_L\, g_b\, \tilde{g}_R^{-1}, \qquad \tilde{g}_L = e^{T_L^b}, \quad \tilde{g}_R = e^{T_R^b}, \tag{5.83}$$

with the generators of H given as

$$T_L^b = \frac{1}{2}\sigma_3, \qquad T_R^b = -\delta\frac{1}{2}\sigma_3. \tag{5.84}$$

The parameter δ is what gives the deformation from the charged 2D black hole ($\delta = 1$). The fermionic sector is given after bosonisation by the $SO(2)$ gauged WZNW model. The group elements g_f are parameterised by

$$g_f = e^{\Phi \frac{i\sigma_2}{\sqrt{2}}} = \begin{pmatrix} \cos\frac{\Phi}{\sqrt{2}} & \sin\frac{\Phi}{\sqrt{2}} \\ -\sin\frac{\Phi}{\sqrt{2}} & \cos\frac{\Phi}{\sqrt{2}} \end{pmatrix} \in SO(2), \tag{5.85}$$

and the gauging in this sector is given by the generators

$$T_L^f = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & Q \\ -Q & 0 \end{pmatrix}, \qquad T_R^f = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \delta \\ -\delta & 0 \end{pmatrix}. \tag{5.86}$$

Note that the right generators T_R^f in the fermionic sector are determined through supersymmetry by the right generators in T_R^b in the bosonic sector.

For this gauging to be non-anomalous, it is necessary that the following *anomaly cancellation condition* is satisfied,

$$k(\delta^2 - 1) - 2(Q^2 - \delta^2) = 0. \tag{5.87}$$

As in section 5.3.4, we will choose the unitary gauge where $t_L = 0$, and write $t_R = t$. Following the steps described in previous sections, and using the notation (5.32), we find

$$\begin{aligned} L_t &= -\frac{1}{2}(k-2)\delta, & L_\Phi &= \delta, \\ R_t &= -\frac{1}{2}(k-2)\cosh(\sigma), & R_\Phi &= -Q, \end{aligned} \tag{5.88}$$

$$G_- = 1, \qquad G_+ = \delta^2, \tag{5.89}$$

$$\begin{aligned} M &= -\frac{1}{2}(k-2)(1+\delta\cosh\sigma) + Q(\delta - Q), \\ \widetilde{M} &= -\frac{1}{2}(k-2)(\delta + \cosh\sigma)\delta + \delta(Q-\delta). \end{aligned} \tag{5.90}$$

This gives the following components of the matrix A_{ij}:

$$\begin{aligned} A_{11} &= 1, \\ A_{12} &= -\frac{1}{2}(k-2)(1+\delta\cosh\sigma) + Q(\delta - Q) - 1, \\ A_{21} &= -\frac{1}{2}(k-2)(\delta+\cosh\sigma)\delta + \delta(Q-\delta) - \delta^2, \\ A_{22} &= \delta^2. \end{aligned} \tag{5.91}$$

The determinant is $\det A = \frac{\delta^2}{4}\Delta$, where

$$\Delta = \big[(k-2)\cosh\sigma + (k+2)\delta - 2Q + 2\big] \times \big[(k-2)\cosh\sigma + (k+2)\delta - 2Q - 2\big]. \tag{5.92}$$

The non-zero fields before re-fermionisation turn out to be:

$$\begin{aligned}
\mathcal{G}_{\sigma\sigma} &= \frac{k-2}{2}, \\
\mathcal{G}_{tt} &= -\frac{k-2}{2}\frac{k(k-2)\cosh^2\sigma - (k+2)^2\delta^2 + 4(k+2)Q\delta - 4Q^2 + 2k}{\Delta(\sigma)}, \\
\mathcal{G}_{\Phi\Phi} &= (k-2)(k+2)\frac{D(\sigma)}{\Delta(\sigma)}, \\
\mathcal{G}_{t\Phi} &= (k-2)\frac{(k-2)\cosh^2\sigma + (k+2)(\delta+Q)\cosh\sigma - (k+2)Q\delta + 2(Q^2+1)}{\Delta(\sigma)}, \\
\mathcal{B}_{t\Phi} &= (k-2)\frac{(\cosh\sigma + Q)\big((k-2)\cosh\sigma + (k+2)\delta - 2Q\big)}{\Delta(\sigma)}, \\
e^{2\hat{\Phi}^{3d}} &= \Delta^{-\frac{1}{2}},
\end{aligned} \tag{5.93}$$

where

$$D(\sigma) = (\cosh\sigma + \delta)^2 - \frac{4}{k+2}(\cosh^2\sigma - 1). \tag{5.94}$$

Next, we have to consider modifications coming from the re-fermionisation. Taking those into account as discussed above, we find the gauge fields

$$\begin{aligned}
A_t &= \frac{(k-2)\cosh^2\sigma + (k+2)(\delta+Q)\cosh\sigma - (k+2)Q\delta + 2(Q^2+1)}{(k+2)D(\sigma)}, \\
B_t &= \frac{(\cosh\sigma + Q)\big((k-2)\cosh\sigma + (k+2)\delta - 2Q\big)}{(k+2)D(\sigma)},
\end{aligned} \tag{5.95}$$

and the exact metric

$$ds^2 = \frac{k-2}{2}\Big[d\sigma^2 - \frac{\cosh^2\sigma - 1}{D(\sigma)}dt^2\Big], \tag{5.96}$$

and the exact dilaton

$$e^{-\hat{\Phi}} = \big[(k-2)(k+2)D(\sigma)\big]^{-\frac{1}{2}}. \tag{5.97}$$

The antisymmetric B-field vanishes in in this model.

For $\delta = 1$ the metric becomes

$$ds^2 = \frac{1}{2}(k-2)\Big[-\Big(\frac{-4}{k+2} + \frac{\cosh\sigma + 1}{\cosh\sigma - 1}\Big)^{-1}dt^2 + d\sigma^2\Big], \tag{5.98}$$

which is very similar to the black hole in section 5.3.4. The only difference is the k dependent term

$$\frac{4}{k+2} \neq \frac{2}{k}. \tag{5.99}$$

But even though $\delta = 1$, the models are not the same, since one is purely bosonic, and the other is a heterotic model including fermions. It is perhaps more surprising that the metrics turn out so similar.

Notice that the fields in eq. (5.93) before re-fermionisation are divergent for $\Delta = 0$, while after re-fermionisation, the divergence is shifted to where $D = 0$. This is a generic feature, and is crucial to make the spacetime fields have a sensible behaviour.

The parametrisation (5.82) of $SL(2, \mathbb{R})$ chosen here gives us the patch of the spacetime corresponding to the region outside the horizon at $\sigma = 0$. However, the solution is easily extended by writing $\cosh \sigma = x$, and allowing x to take all real values. I will come back to this in the next chapter.

Chapter 6

High energy corrections in a stringy Taub-NUT spacetime

In this chapter I study an exact model of string theory propagating in a spacetime containing regions with closed timelike curves (CTCs) separated from a finite cosmological region bounded by a Big Bang and a Big Crunch. The model is a non-trivial embedding of the Taub-NUT geometry into heterotic string theory with a full conformal field theory (CFT) definition, discovered over a decade ago as a heterotic coset model. Having a CFT definition makes this an excellent laboratory for the study of the stringy fate of CTCs, the Taub cosmology, and the Milne/Misner-type chronology horizon which separates them. In an effort to uncover the role of stringy corrections to such geometries, I calculate the complete set of α' corrections to the geometry, and observe that the key features of Taub-NUT persist in the exact theory, together with the emergence of a region of space with Euclidean signature bounded by timelike curvature singularities. Although such remarks are premature, their persistence in the exact geometry suggests that string theory is able to make physical sense of the Milne/Misner singularities and the CTCs, despite their pathological character in General Relativity. This may also support the possibility that CTCs may be viable in some physical situations, and may be a natural ingredient in pre-Big-Bang cosmological scenarios.

The results of this chapter has been submitted for publication [2], with my supervisor Prof. Clifford V. Johnson as co-author.

6.1 Introduction and Motivation

The Taub-NUT spacetime [121,122] is an interesting one. We can write a metric for it as follows:

$$ds^2 = -f_1(dt - l\cos\theta d\phi)^2 + f_1^{-1}dr^2 + (r^2+l^2)(d\theta^2 + \sin^2\theta d\phi^2) \ , \tag{6.1}$$

where

$$f_1 = 1 - 2\frac{Mr+l^2}{r^2+l^2} \ . \tag{6.2}$$

The angles θ and ϕ are the standard angles parameterising an S^2 with ranges $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$. In addition to simple time translation invariance, the metric has an $SO(3)$ invariance acting as rotations on the S^2. To preserve $d\xi = dt - l\cos\theta d\phi$, a time translation must also accompany a general rotation. This makes t periodic with period $4l\pi$, which can be deduced by asking for there to be no conical singularities in the north or south poles. The coordinate t is fibred over the S^2 making a squashed S^3, and the full invariance is under an $SU(2)$ action on this space.

There are two very different regions of this spacetime, as we move in r, distinguished by the sign of $f_1(r)$. The regions are separated by the loci (with S^3 topology)

$$r_\pm = M \pm \sqrt{M^2+l^2} \ , \tag{6.3}$$

where f_1 vanishes. They are, in a sense, horizons. The metric is singular there, and although there exist extensions, the nature of these is subtle in General Relativity (for a review, see ref. [123]). One of the things which we will see in detail later is the fact that the string theory provides an extremely natural extension.

The region $r_- < r < r_+$ has $f_1(r) < 0$. The coordinate r plays the role of time, and the geometry changes as a function of time. This is the "Taub" cosmology, and spatial slices have the topology of an S^3. The volume of the universe begins at $r = r_-$ at zero, it expands to a maximum value, and then contracts to zero again at $r = r_+$. This is a classical "Big Bang" followed by a classical "Big Crunch".

On either side of this Taub region, we have $f_1(r) > 0$. The coordinate t plays the role of time, and there is a static spatial geometry. But since t is periodic, it is threaded by closed timelike curves. Constant radial slices have the topology of an S^3 where the time is a circle fibred over the S^2. These regions are called the "NUT" regions.

It is fascinating to note that the Taub and NUT regions are connected. There are geodesics which can pass from one region to another, and analytic extensions of the metric can be written down [123]. The geometry is therefore interesting, since it presents itself as a laboratory for the study of a cosmology which naturally comes capped with regions containing CTCs. Classical physics would seem to suggest that we can begin within the cosmological region and after waiting a finite time, find that the universe contained closed timelike loops.

It is an extremely natural question to ask whether or not this is an artefact of classical physics, a failure of General Relativity to protect itself from the apparent pathologies with which such time machines seem to be afflicted. This leads to a closer examination of the neighbourhood of the loci $f_1(r) = 0$ located at $r = r_\pm$, which are the "chronology horizons". For small $\tau = r - r_-$, we see that $f_1 = -c\tau$, where c is a constant, and we get for the (τ, ξ) plane:

$$ds^2 = -(c\tau)^{-1}d\tau^2 + c\tau d\xi^2 \ , \tag{6.4}$$

which is the metric of a two dimensional version of the "Milne" Universe, or "Misner space" [124]. It is fibred over the S^2.

There is an early study of cosmological singularities of this type in a semi-classical quantum treatment, reported on in ref. [125]. There, the vacuum stress-energy tensor for a conformally coupled scalar field in the background is computed, and it diverges at $\tau = 0$. This is taken by some as an encouraging sign that a full theory of quantum gravity might show that the geometry is unstable to matter fluctuations and the appropriate back–reaction should give a geometry which is modified at the boundaries between the Taub and NUT regions. In fact, this is the basis of the "chronology protection conjecture" of ref. [74], which suggests (using Taub-NUT as a one of its key examples) that the full physics will conspire to forbid the creation of CTCs in a spacetime that does not already have them present, *i.e.*, the Misner geometry of the chronology horizon is destroyed and replaced by a non-traversable region. (However, even staying within Relativity, there are many who take an alternative view, by *e.g.*, showing that a non-divergent stress tensor can be obtained by computing in a different vacuum, thus calling into the question the need for such a conjecture. See for example, refs. [73, 126–134] and for a recent stringy example, see ref. [135].) The expectations of a full theory of quantum gravity in this regard are (at least) two–fold: (1) It should prescribe exactly what types of matter propagate in the geometry, and; (2) It should give a prescription for exactly how the geometry is modified, incorporating any back-reaction of the matter on the geometry in a self-consistent way.

Since the papers of ref. [74, 125], a lot has happened in fundamental physics. In particular, it is much clearer that we do in fact have a quantum theory of gravity on the market, that should allow us to study the questions above[1]. Of course, I am referring to string theory (including its not yet fully defined non-perturbative completion in terms of M-theory). While the theory has yet to be developed to the point where we can address the physics of spacetime backgrounds as thoroughly as possible, there are many questions which we can ask of the theory, and in certain special cases, we can study certain spacetime backgrounds in some detail.

In fact, as I will review in the next section, the Taub-NUT spacetime can be embedded into string theory in a way that allows its most important features to be studied in a very controlled laboratory, an *exact* conformal field theory [115]. It is therefore not just accessible as a solution to the leading order in an expansion in small α' (the inverse string tension), but to all orders and beyond. Leading order captures only the physics of the massless modes of the string, (the low energy limit) and so any back-reaction affecting the geometry via high-energy effects cannot be studied in this limit. With the full conformal field theory we can in principle extract the complete geometry, including all the effects of the infinite tower of massive string states that propagate in it. I do this in the present chapter and extract the fully corrected geometry. We will see that the key features of the geometry *survive* to all orders in α', even though placed in a string theory setting without any special properties to forbid corrections. This result means that a large family of high energy effects which *could* have modified the geometry are suppressed by the full string theory. The strings seem to propagate in this apparently pathological geometry with no trouble at all. It is of course possible that the new geometry we find is unstable to the presence of a test particle or string, but this type of effect does not show up in the CFT in this computation. Such test-particle effects are important to study[2] in order to understand the complete fate of the geometry by studying its stability against fluctuations. The work reported here yields the fully corrected geometry in which such probe computations should be carried out in this context. More properly, the probe computation should be done in the full conformal field theory, in order to allow the string theory to respond fully to the perturbation. The conformal field theory discussed here is a complete laboratory for such studies, and as it describes the Taub-NUT geometry, it provides the most natural stringy analogue of this classic geometry within which to answer many interesting questions. There are a number of other interesting conformal field theories (and studies thereof) which have been

[1]Leaving aside the question of CTCs, cosmological singularities of Misner type have recently become relevant in the context of cosmologies inspired by string- (and M-) theory. See for example ref. [136].

[2]They have been studied for the leading order geometry in its form as an orbifold of Minkowski space by a Lorentz boost [137–141].

presented, which at low energy describe geometries which although are not Taub-NUT spacetimes, do share many of the key features in local patches. Some of them are listed in refs. [142–157]. Refs. [156, 157] additionally contain useful comments and literature survey. There are also many papers on the properties of string theory in spacetimes with CTCs, such as the BMPV [64] spacetime [65, 76, 77, 80, 158–166] (see also chapter 4) and the Gödel [167] spacetime [69, 78, 79, 168–171].

In section 6.2, I recall the stringy Taub-NUT metric discovered in ref. [115], and write it in a new coordinate which gives it a natural extension exhibiting the Taub and NUT regions and their connection *via* Misner space. I also recall the work of refs. [172–174] which demonstrates how to obtain the low energy metric as a stringy embedding by starting with the standard Taub-NUT metric of equation (6.1). It is the "throat" or "near-horizon" region of this spacetime that was discovered in ref. [115], where an exact conformal field theory (a heterotic coset model) can be constructed which encodes the full stringy corrections. Then I review the conformal field theory construction in sections 6.3.1, where the Lagrangian definition is written down. Happily, the extension of the throat geometry presented in section 6.2 (described by the same conformal field theory) contains all the interesting features: the Taub region with its Big-Bang and Big-Crunch cosmology, the NUT regions with their CTCs, and the Misner space behaviour which separates them. Therefore we have a complete string theory laboratory for the study of the properties of Taub-NUT, allowing us to address many of the important questions raised in the Relativity community. For example, questions about the analytic extension from the NUT to the Taub regions are put to rest by the fact that the full conformal field theory supplies a natural extension *via* the structure of $SL(2,\mathbb{R})$ (section 6.2). Further, having the full conformal field theory means that we can construct the α' corrections to the low energy metric, and I do this in section 6.4, capturing *all* of the corrections, after constructing an exact effective action in section 6.3.2. I analyse the exact metric in section 6.4.2, and end the chapter with a discussion in section 6.5, noting that there are many questions that can be answered in this laboratory by direct computation in the fully defined model.

6.2 Stringy Taub-NUT

Taub-NUT spacetime, being an empty-space solution to the Einstein equations, is trivially embedded into string theory with no further work. It satisfies the low-energy equations of motion of any string theory, where the dilaton is set to a constant and all the other background fields are set to zero. This is not sufficient for what I want to do here, since I want to have a means of getting efficient computational access to the stringy corrections to the geometry. We need a new embedding which allows such computational control.

This was achieved some time ago. An exact conformal field theory describing the Taub-NUT spacetime (in a certain "throat" or "near-horizon" limit) was constructed in ref. [115]. This CFT will be described in the next section. The geometry comes with a non-trivial dilaton and antisymmetric tensor field, together with some electric and magnetic fields. The string theory is heterotic string theory. This model is in fact the earliest non-trivial embedding of Taub-NUT into string theory, and uses the heterotic coset model construction in order to define the theory [115, 175–177]. The technique was discovered as a method of naturally defining $(0,2)$ conformal field theories, *i.e.*, backgrounds particularly adapted to yielding minimally supersymmetric vacua of the heterotic string. That aspect will not be relevant here, since I will not tune the model in order to achieve spacetime supersymmetry.

The low-energy metric of the stringy Taub-NUT spacetime was presented in ref. [115] as (in string frame):

$$ds^2 = k\left\{d\sigma^2 - \frac{\cosh^2\sigma - 1}{(\cosh\sigma+\delta)^2}(dt - \lambda\cos\theta d\phi)^2 + d\theta^2 + \sin^2\theta d\phi^2\right\}\ , \tag{6.5}$$

where $0 \le \sigma \le \infty$, $\delta \ge 1$, $\lambda \ge 0$. The dilaton behaves as:

$$\Phi - \Phi_0 = -\frac{1}{2}\ln(\cosh\sigma + \delta)\ , \tag{6.6}$$

and there are other fields which I will discuss later. This is in fact the NUT region of the geometry, and $\sigma = 0$ is a Misner horizon. Note here that the embedding presents a natural analytic extension of this model which recovers the other NUT region and the Taub cosmology as well: Replace $\cosh\sigma$ with the coordinate x:

$$ds^2 = k\left(\frac{dx^2}{x^2-1} - \frac{x^2-1}{(x+\delta)^2}(dt - \lambda\cos\theta d\phi)^2 + d\theta^2 + \sin^2\theta d\phi^2\right)\ , \tag{6.7}$$

with

$$\Phi - \Phi_0 = -\frac{1}{2}\ln(x + \delta)\ , \tag{6.8}$$

where now $-\infty \le x \le +\infty$. The three ranges of interest are $+1 \le x \le +\infty$, ($x = \cosh\sigma$) which is the first NUT region above, $-\infty \le x \le -1$ ($x = -\cosh\sigma$) which is a second NUT region, and $-1 \le x \le +1$ ($x = -\cos\tau$), which a Taub region with a Big Bang at $\tau = 0$ and a Big Crunch at $\tau = \pi$. We shall see shortly that this embedding is very natural from the point of view of string theory, since x is a natural coordinate on the group $SL(2,\mathbb{R})$, which plays a crucial role in defining the complete theory. It is interesting to sketch the behaviour of the function $G_{tt} = F(x) = (1-x^2)/(x+\delta)^2$. This is done in figure 6.1. Note that $F(x)$ vanishes at $x = \pm 1$ and so for $x = 1 - \tau$ where τ is small, the metric of the (τ, ξ) space is:

$$ds^2 = k\left(-(2\tau)^{-1}d\tau^2 + \frac{2\tau}{(1+\delta)^2}d\xi^2\right) , \tag{6.9}$$

which is of Misner form, and so the essential features of the Taub–NUT spacetime persist in this stringy version of the spacetime. Note that, unlike General Relativity's

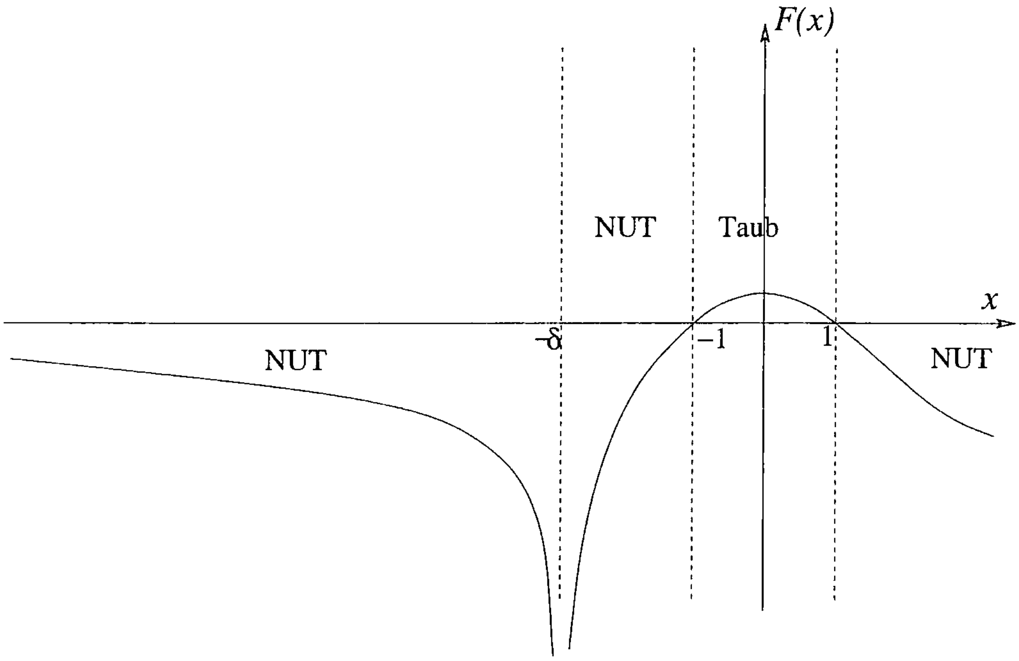


Figure 6.1: The various regions in the stringy Taub-NUT geometry. There are two NUT regions, containing CTCs, and a Taub region. which is a cosmology. Note that there is a curvature singularity in the second NUT region, when $x = -\delta$.

Taub-NUT solution, there is a genuine curvature singularity in the metric, and it is located at $x = -\delta$. The dilaton diverges there, and hence the string theory is strongly coupled at this place, but it is arbitrarily far from the regions of Misner space connecting the Taub and NUT regions, so we need not worry about this locus for the questions of interest here.

Note that the (x,t) plane is fibred over a family of S^2s which have *constant* radius, as opposed to a radius varying with x. This does not mean that we lose key features of the geometry, since *e.g.* in the Taub region, we still have a cosmology in which the universe has S^3 topology, but its volume is controlled entirely by the size of the circle fibre $(dt - \lambda\cos\theta d\phi)$, which ensures that the universe's volume vanishes at the beginning and the end of the cosmology.

The constancy of the S^2s is in fact a feature, not a bug. It allows the geometry to be captured in an exact conformal field theory, as I shall recall in the next section. This geometry is the "near-horizon" limit of a spacetime constructed as confirmation of the statement in ref. [115] that the metric in question is indeed obtainable from the original Taub-NUT metric in a series of steps using the symmetries of the heterotic string theory action [172–174]. This geometry is, in string frame:

$$ds^2 = (a^2+f_2^2)\left\{-\frac{f_1}{f_2^2}(dt+(\rho+1)l\cos\theta d\phi)^2 + f_1^{-1}dr^2 + (r^2+l^2)(d\theta^2+\sin^2\theta d\phi^2)\right\} , \tag{6.10}$$

where f_1 is as before, $\rho^2 \geq 1$ and

$$f_2 = 1 + (\rho-1)\frac{Mr+l^2}{r^2+l^2} , \qquad \text{and} \qquad a = (\rho-1)l\frac{r-M}{r^2+l^2} . \tag{6.11}$$

This metric has the full asymptotically flat part of the geometry and connects smoothly onto the throat region, which develops in an "extremal" limit (analogous to that taken for charged black holes). This is shown schematically in figure 6.2. The metric (6.5) is obtained from it in the extremal limit $\rho \to \infty, M \to 0, \lambda \to 0$, where $m = \rho M$ and $\ell = \rho l$ are held finite. The limit is taken in the neighbourhood of $f_1 = 0$, and σ is the scaled coordinate parameterising r in that region. The parameters of metric (6.5) are recovered as: $\lambda = l/m$ and $\delta^2 = 1 + l^2/M^2$.

The metric (6.10), and other fields of the solution which are not displayed here, are given a stringy embedding as follows (the details are in refs. [172–174]). We start from the metric (6.1) viewed as a solution of heterotic string theory. Then, an $O(1,1)$ boost (a subgroup of the large group of perturbative non-compact symmetries possessed by the heterotic theory) is used to generate a new solution, mixing the t direction with a $U(1)$ gauge direction. This generates a gauge field A_t, a non-trivial dilaton, and since there is a coupling of t to ϕ in the original metric, a gauge field A_ϕ and an antisymmetric tensor background $B_{t\phi}$. So the solution has electric and magnetic charges under a $U(1)$ of the heterotic string, and non-trivial axion and dilaton charge. We will not need the forms of the fields here. It turns out that the dilaton has a behaviour which is "electric" in a sense inherited from the behaviour of charged dilaton black holes: It decreases as one approaches the horizon. Such holes do not support the development of throats in the string frame metric, but their "magnetic" cousins, where the dilaton has the opposite behaviour, do support throats. (In fact, an exact conformal field theory can be written for pure magnetic dilaton black holes in four dimensions [178], and it can be realised as a heterotic

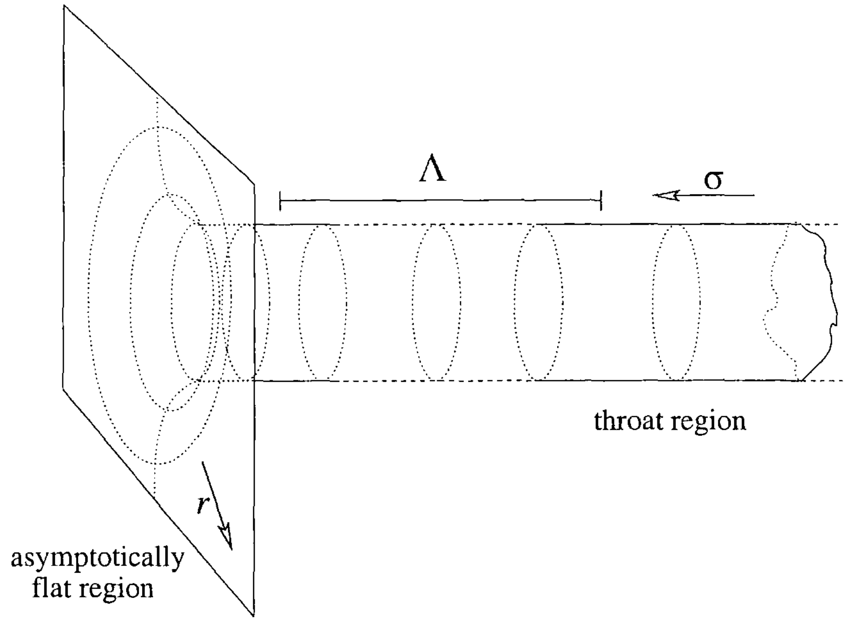


Figure 6.2: A schematic showing the asymptotically flat region connected to the throat region located near the horizon at extremality. In the extremal limit, the typical measure, Λ of the distance from a point on the outside to a point near the horizon region diverges logarithmically, and the throat region is infinitely long. The coordinate σ is used for the exact throat region in low-energy metric (6.5), while r is the coordinate for the general low energy metric (6.10).

coset model as well [115].) Using the $SL(2,\mathbb{R})$ S-duality of the four dimensional effective action of the heterotic string, which combines an electric-magnetic duality with an inversion of the axi-dilaton field $\tau = a + ie^{-\Phi}$, a solution with "magnetic" character can be made [172,173], which supports a throat in the string frame metric. This is the solution whose metric is displayed in equation (6.10).

So in summary, there is an embedding of the Taub-NUT solution of General Relativity into heterotic string theory which preserves all of the interesting features: the NUT regions containing CTCs, and the Taub region with its Big Bang and Big Crunch cosmology, and (crucially) the Misner regions connecting them. There is a throat part of the geometry which decouples from the asymptotically flat region in an extremal limit, but which captures all of the features of the Taub-NUT geometry of interest to us here.

The next thing we need to recall is that this throat geometry arises as the low energy limit of a complete description in terms of a conformal field theory [115].

6.3 Exact Conformal Field Theory

6.3.1 The Definition

The heterotic coset model technique (see section 5.4) was presented in ref. [115], and one of the examples of the application of the method was the model in question, from which the low energy metric in equation (6.7) was derived, for $x = \cosh\sigma$. The other regions that have been presented here (making up $-\infty \leq x \leq 1$) are easily obtained from the same conformal field theory by choosing different coordinate patches in the parent model, as we shall see.

By way of example, let me simply present the model relevant to the study here [115]. The group in question is $SL(2,\mathbb{R}) \times SU(2)$, and the group elements are denoted g_1 and g_2 respectively. Let the levels of the models be denoted k_1 and k_2, respectively. We are interested in a $U(1)_A \times U(1)_B$ subgroup (A and B are just means of distinguishing them) which acts as follows:

$$U(1)_A \times U(1)_B : \begin{cases} g_1 & \longrightarrow \quad e^{\epsilon_A\sigma_3/2}g_1e^{(\delta\epsilon_A+\lambda\epsilon_B)\sigma_3/2} \\ g_2 & \longrightarrow \quad g_2e^{i\epsilon_B\sigma_3/2} \end{cases} \tag{6.12}$$

Notice that there is a whole global $SU(2)_L$ of the original $SU(2)_L \times SU(2)_R$ untouched. This is a deliberate choice to give a model with spacetime $SU(2)$ invariance (rotations) in the end. With that, and the other asymmetry introduced by the presence of λ and δ, the gauging is very anomalous. Once right-moving supersymmetry fermions are introduced, the anomalies are proportional to $-k_1(1-\delta^2)+2\delta^2$ from the AA sector, $k_1\delta\lambda+2\delta\lambda$ from the AB sector, and $k_2+k_1\lambda^2+2(1+\lambda^2)$ from the BB sector. The k-independent parts come from the fermions. Next, four left-moving fermions are introduced. Two are given charges $Q_{A,B}$ under $U(1)_{A,B}$ and the other two are given charges $P_{A,B}$. Their anomalies are $-2(Q_A^2+P_A^2)$, $-2(Q_AQ_B+P_AP_B)$, and $-2(Q_B^2+P_B^2)$, respectively, from the various sectors AA, AB, BB. So we can achieve an anomaly-free model by asking that:

$$\begin{aligned} -k_1(1-\delta^2) &= 2(Q_A^2+P_A^2-\delta^2) \\ k_1\delta\lambda &= 2(Q_AQ_B+P_AP_B-\delta\lambda) \\ k_2+k_1\lambda^2 &= 2(Q_B^2+P_B^2-(1+\lambda^2)) \ . \end{aligned} \tag{6.13}$$

In ref. [115] the observation was made that the stringy solution (6.5) can be obtained from the basic Taub-NUT solution (6.1), and this was verified using the solution-generating techniques summarised in the previous section. It is a highly non-trivial check on the consistency of the model to note that in those calculations, the charges in the throat metric turn out to be given in terms of the parameters M, l and ρ in such a way that they satisfy the anomaly equations above, in the large k limit (which is appropriate to low-energy). See ref. [172].

The central charge of this four dimensional model is:

$$c = \frac{3k_1}{k_1 - 2} + \frac{3k_2}{k_2 + 2} \ , \tag{6.14}$$

where the -2 from gauging is cancelled by the $+2$ from four bosons on the left and right. We can ask that this be equal to 6, as is appropriate for a four dimensional model, tensoring with another conformal field theory to make up the internal sector, as desired[3]. The result is that $k_1 = k_2 + 4$.

In ref. [115], the metric for the throat region was discovered by working in the low energy limit where k_1 and k_2 are large, and denoted simply as k. In this chapter, I will study the model beyond this large k (low energy) approximation and derive the geometry which is correct to all orders in the $\alpha' \sim 1/k$ expansion.

6.3.2 Writing The Full Action

The $G = SL(2,\mathbb{R}) \times SU(2)$ WZNW model is given by:

$$S(g_1, g_2) = -k_1 I_{WZNW}(g_1) + k_2 I_{WZNW}(g_2) \ , \tag{6.15}$$

where the WZNW action I_{WZNW} is defined in equation (5.3). The group valued fields $g_1(z,\bar{z}) \in SL(2,\mathbb{R})$ and $g_2(z,\bar{z}) \in SU(2)$ map the worldsheet Σ with coordinates $(z,\bar{z})$ into the group $SL(2,\mathbb{R}) \times SU(2)$. Part of the model is defined by reference to an auxiliary spacetime $\mathcal{B}$, whose boundary is Σ, with coordinates σ^a. The action $\Gamma(g)$ is simply the pull-back of the $G_L \times G_R$ invariant three-form on G.

With reference to the $U(1)_A \times U(1)_B$ action chosen in equation (6.12), the gauge fields are introduced with the action:

$$\begin{aligned} S(g_1, g_2, A) = \frac{k_1}{8\pi} \int d^2z \Big\{ & -2(\delta A^A_{\bar{z}} + \lambda A^B_{\bar{z}}) \mathrm{Tr}[\sigma_3 g_1^{-1} \partial_z g_1] - 2A^A_z \mathrm{Tr}[\sigma_3 \partial_{\bar{z}} g_1 g_1^{-1}] \\ & + A^A_z A^A_{\bar{z}} (1 + \delta^2 + \delta \mathrm{Tr}[\sigma_3 g_1 \sigma_3 g_1^{-1}]) + \lambda^2 A^B_z A^B_{\bar{z}} \\ & + \lambda\delta A^A_z A^B_{\bar{z}} + A^B_z A^A_{\bar{z}} (\lambda\delta + \lambda \mathrm{Tr}[\sigma_3 g_1 \sigma_3 g_1^{-1}]) \Big\} \\ + \frac{k_2}{8\pi} \int d^2a \, \{ & 2i A^B_{\bar{z}} \mathrm{Tr}[\sigma_3 g_2^{-1} \partial_z g_2] + A^B_z A_{\bar{z}} B \} \ . \end{aligned} \tag{6.16}$$

[3] Actually, we can also choose other values of c, and adjust the internal theory appropriately.

The gauge generators have been written as

$$t^{(1)}_{A,R} = -\delta\frac{\sigma_3}{2}, \quad t^{(1)}_{A,L} = \frac{\sigma_3}{2}, \quad t^{(1)}_{B,R} = -\lambda\frac{\sigma_3}{2}, \quad t^{(2)}_{B,R} = -i\frac{\sigma_3}{2} \ . \tag{6.17}$$

The anomaly under variation $\delta A_a^{A(B)} = \partial_a \epsilon_{A(B)}$ can be written as:

$$\mathcal{A}_{ab} = \frac{1}{4\pi}\mathrm{Tr}[t_{a,L}t_{b,L} - t_{a,R}t_{b,R}]\epsilon_a \int d^2z F^b_{z\bar{z}} \ , \tag{6.18}$$

(no sum on a, b) and I have defined $\mathrm{Tr} = -k_1\mathrm{Tr}_1 + k_2\mathrm{Tr}_2$. The right-moving fermions have an action:

$$I^F_R = \frac{i}{4\pi}\int d^2z \mathrm{Tr}(\Psi_R \mathcal{D}_z \Psi_R) \ , \tag{6.19}$$

where Ψ_R takes values in the orthogonal complement of the Lie algebra of $U(1)_A \times U(1)_B$, (so there are four right-movers, in fact) and

$$\mathcal{D}_z\Psi_R = \partial_z\Psi_R - \sum_a A^a_{\bar{z}}[t_{a,R}, \Psi_R] \ . \tag{6.20}$$

The four left-moving fermions have action:

$$\begin{aligned} I^F_L = &-\frac{ik_1}{4\pi}\int d^2z \left\{\lambda^1_L[\partial_z + Q_A A^A_z + Q_B A^B_z]\lambda^2_L\right\} \\ &+\frac{ik_2}{4\pi}\int d^2z \left\{\lambda^3_L[\partial_z + P_A A^A_z + P_B A^B_z]\lambda^4_L\right\} \ . \end{aligned} \tag{6.21}$$

Under the gauge transformation $\delta A_a^{A(B)} = \partial_a \epsilon_{A(B)}$, these two sets of fermion actions yield the anomalies discussed earlier, but at one-loop, while the WZNW model displays its anomalies classically. It is therefore difficult to work with the model in computing a number of properties. In particular, in working out the effective spacetime fields it is useful to integrate out the gauge fields. It is hard to take into account the effects of the successful anomaly cancellation if part of them are quantum and part classical. The way around this awkward state of affairs [115] is to bosonise the fermions. The anomalies of the fermions then appear as classical anomalies of the action. The bosonised action is:

$$\begin{aligned} I_B = \frac{1}{4\pi}\int d^2z\Big\{&[\partial_z\Phi_2 - P_A A^A_{\bar{z}} - (P_B+1)A^B_{\bar{z}}]^2 + [\partial_z\Phi_1 - (Q_B+\lambda)A^B_{\bar{z}} - (Q_A+\delta)A^A_{\bar{z}}]^2 \\ &- \Phi_1[(Q_B-\lambda)F^B_{z\bar{z}} + (Q_A-\delta)F^A_{z\bar{z}}] - \Phi_2[(P_B-1)F^B_{z\bar{z}} + P_A F^A_{z\bar{z}}] \\ &+ [A^A_{\bar{z}}A^B_z - A^A_z A^B_{\bar{z}}][\delta Q_B - \lambda Q_A - P_A]\Big\} \ , \end{aligned} \tag{6.22}$$

which under variations:

$$\delta A_a^{A(B)} = \partial_a \epsilon_{A(B)} \ , \quad \delta\Phi_1 = (Q_A+\delta)\epsilon_A + (Q_B+\lambda)\epsilon_B \ , \quad \delta\Phi_2 = P_A\epsilon_A + (P_B+1)\epsilon_B \ , \tag{6.23}$$

manifestly reproduces the anomalies presented earlier.

6.3.3 Extracting the Low Energy Metric

At this stage, it is possible to proceed to derive the background fields at leading order by starting with the Lagrangian definition given in the previous section and integrating out the gauge fields, exploiting the fact that they appear quadratically in the action. As these fields are fully quantum fields, this procedure is only going to produce a result which is correct at leading order in the $1/k$ expansion, where k is large. This is because we are using their equations of motion to replace them in the action, and neglecting their quantum fluctuations. Before turning to how to go beyond that, let us note that there is an important subtlety even in the derivation of the leading order metric. This is not an issue for coset models that are not built in this particular heterotic manner, and so is a novelty that cannot be ignored.

The coordinates we shall use for $SL(2,\mathbb{R})$ and $SU(2)$ are:

$$g_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{t_+/2}(x+1)^{1/2} & e^{t_-/2}(x-1)^{1/2} \\ e^{-t_-/2}(x-1)^{1/2} & e^{-t_+/2}(x+1)^{1/2} \end{pmatrix} \ , \tag{6.24}$$

where $t_\pm = t_L \pm t_R$, and $-\infty \leq t_R, t_L, x \leq \infty$, and the Euler angles

$$g_2 = \begin{pmatrix} e^{i\phi_+/2}\cos\frac{\theta}{2} & e^{i\phi_-/2}\sin\frac{\theta}{2} \\ -e^{-i\phi_-/2}\sin\frac{\theta}{2} & e^{-i\phi_+/2}\cos\frac{\theta}{2} \end{pmatrix} \ , \tag{6.25}$$

where $\phi_\pm = \phi \pm \psi$, $0 \leq \theta \leq \pi$, $0 \leq \psi \leq 4\pi$, and $0 \leq \phi \leq 2\pi$. Note that the full range of x is available here, while remaining in $SL(2,\mathbb{R})$. In ref. [115] (and in the examples I discussed in chapter 5), the range $x = \cosh\sigma \geq 1$ was used. The larger range reveals the connection to the Taub and the other NUT region. This extension is very naturally inherited from the $SL(2,\mathbb{R})$ embedding[4].

Before integrating out the gauge fields, let us fix to the gauge:

$$t_L = 0 \ , \quad \psi = \pm\phi \ , \tag{6.26}$$

[4]See ref. [149] for a discussion of how an $SL(2,\mathbb{R})$ structure also provides a natural extension for the discussion of wavefunctions in related spacetimes.

where + refers to the north pole on the S^2 parametrised by (θ, ϕ) and $-$ refers to the south pole. In the following I will write $t_R = t$.

I will shortly turn to the problem of deriving the *exact* spacetime fields, as discussed in chapter 5, but let me first summarise how the leading order expressions were found in ref. [115].

Having fixed the gauge, and integrated out the gauge fields, we can read off various spacetime fields from the resulting nonlinear sigma model by examining terms of the form $C_{MN}\partial_z X^M \partial_{\bar{z}} X^N$, where here X^M, is a place holder for any worldsheet field, and M denotes which field is present. When M, N are such that $X^M X^N$ run over the set of fields t, x, θ, ϕ, then the symmetric parts of C_{MN} give a metric that was called $G^0_{\mu\nu}$, and the antisymmetric parts give the antisymmetric tensor potential $B_{\mu\nu}$. When M, N are such that X^M is one of the bosonised fermions and X^N is one of t, x, θ, ϕ, the C_{MN} is a spacetime gauge potential, either from the (1) or the (2) sector: $A^{(1,2)}_\mu$.

It was noted that $G^0_{\mu\nu}$ is *not* the correct spacetime metric at this order. This is a crucial point. The anomaly cancellation requirement means that the contribution from the left-movers leads to a significant modification to the naive metric. The most efficient way of seeing how it is modified is to re-fermionise the bosons, using as many symmetries as we can to help in deducing the normalisation of the precise couplings. After some work, it transpires that the correct metric (to leading order) is:

$$G_{\mu\nu} = G^0_{\mu\nu} - \frac{1}{2k}[A^1_\mu A^1_\nu + A^2_\mu A^2_\nu]. \tag{6.27}$$

Because $A \sim Q$, and since $Q \sim \sqrt{k}$ as is apparent from the anomaly equations (6.13), we can see that this gives a non-trivial correction to the metric we would read off naively. This is the clearest sign that these heterotic coset models are quite different from coset models that have commonly been used to make heterotic string backgrounds by tensoring together ordinary cosets. In those cases, typically $A \sim Q \sim 1$ and so at large k, the correction is negligible.

This sets the scene for what we will have to do when we have constructed the exact effective nonlinear sigma model. We will again need to correct the naive metric in a way which generalises equation (6.27), in order to get the right spacetime metric.

6.4 Exact spacetime fields

Let me now turn to the question of how to extract the exact spacetime fields. I shall exploit the fact [115, 175, 176] that the heterotic coset model, in its bosonised form where all the anomalies are classical, can be thought of as an asymmetrically gauged

WZNW model for G/H supplemented by another asymmetrically gauged WZNW model for $SO(\dim G - \dim H)/H$, representing the fermions. The subsequent calculations will follow the procedure discussed in chapter 5. First, we carry out a change of variables that enables us to write the whole model as a set of decoupled WZNW models, then transform to the effective action, and subsequently re-write it back in the original variables to see what new terms the effective action supplies us with. Next, we integrate out the gauge fields and –crucially– correctly re-fermionise the bosons to read off the spacetime fields. These computations are summarised in the following subsection.

6.4.1 Computations

The parametrisation of the gauge groups $SL(2,\mathbb{R})$, and $SU(2)$ have already been given in equations (6.24) and (6.25). The set of generators $t^{(1)}$ and $t^{(2)}$ of the gauge group $H = U(1)_A \times U(1)_B$ have been given in equation (6.17) when acting on the $H \subset SL(2,\mathbb{R})$ and $H \subset SU(2)$ parts respectively. Recall also that this gauging leaves the global $SU(2)_L$ symmetry untouched, and so it will survive as a global symmetry of the final model; the $SU(2)$ rotational invariance of Taub-NUT. Finally, introduce the generators of H when acting on the fermionic part, $H \subset SO(4)$:

$$t^{(f)}_{A,L} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -Q_A & & \\ Q_A & 0 & & \\ & & 0 & P_A \\ & & -P_A & 0 \end{pmatrix}, \quad t^{(f)}_{A,R} = -\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -\delta & & \\ \delta & 0 & & \\ & & 0 & 0 \\ & & 0 & 0 \end{pmatrix},$$

$$t^{(f)}_{B,L} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -Q_B & & \\ Q_B & 0 & & \\ & & 0 & P_B \\ & & -P_B & 0 \end{pmatrix}, \quad t^{(f)}_{B,R} = -\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -\lambda & & \\ \lambda & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix}. \tag{6.28}$$

Note that the $t^{(f)}_R$ are fixed by $(0,1)$ worldsheet supersymmetry, while in the $t^{(f)}_L$, the $Q_{A,B}$ and $P_{A,B}$ are chosen to cancel the anomaly *via* equation (6.13). The group element $g_f \in SO(4)$, corresponding to the bosonised fermions, is parameterised as:

$$g_f = \exp\left\{\begin{pmatrix} \Phi_1 \frac{i\sigma_2}{\sqrt{2}} & \\ & -\Phi_2 \frac{i\sigma_2}{\sqrt{2}} \end{pmatrix}\right\} = \begin{pmatrix} \cos\frac{\Phi_1}{\sqrt{2}} & \sin\frac{\Phi_1}{\sqrt{2}} & & \\ -\sin\frac{\Phi_1}{\sqrt{2}} & \cos\frac{\Phi_1}{\sqrt{2}} & & \\ & & \cos\frac{\Phi_2}{\sqrt{2}} & -\sin\frac{\Phi_2}{\sqrt{2}} \\ & & \sin\frac{\Phi_2}{\sqrt{2}} & \cos\frac{\Phi_2}{\sqrt{2}} \end{pmatrix}, \tag{6.29}$$

where Φ_1 and Φ_2 are 2π periodic.

Note that we have effectively gauge-fixed the fermionic sector by only writing enough fields to fill out an $SO(2) \times SO(2)$ subgroup of the $SO(4)$. This also anticipates that we will choose the gauge given in (6.26) so as to remove two fields out of the six given by fully parameterising the $SL(2,\mathbb{R}) \times SU(2)$, therefore retaining Φ_1 and ϕ_2 in the final model.

To find the coefficients C_{MN} we now have to compute the group manifold metric g_{MN} and the vectors L_M and R_M. We also have to compute the matrix A_{ij} and find its inverse. This is all relatively straightforward and the details, involving a number of rather messy expressions, are left out. Having completed this task, we end up with an action of the form

$$S = \int d^2z(\mathcal{G}_{MN} + \mathcal{B}_{MN})\partial X^M \bar{\partial} X^N, \tag{6.30}$$

where $\mathcal{G}_{MN}$ is symmetric, and $\mathcal{B}_{MN}$ is antisymmetric. The fields X^M can be $t, x, \theta, \phi, \Phi_1, \Phi_2$, so this is effectively a bosonic six-dimensional theory as it is written.

We must then worry about the effects of re-fermionisation. How this is done was discussed in general in section 5.4.3, and it is straightforward to apply those ideas to the present case. Carrying out the computations, we find that the final expression for the exact metric simplifies in a remarkable way to the following (using equation (6.14) to write $k_1 = k$, $k_2 = k-4$):

$$\begin{aligned} ds^2 &= G_{\mu\nu}dX^\mu dX^\nu \\ &= (k-2)\left\{\frac{dx^2}{x^2-1} - \frac{x^2-1}{D(x)}(dt + 2\lambda A_\phi^M d\phi)^2 + d\theta^2 + \sin^2\theta d\phi^2\right\}, \end{aligned} \tag{6.31}$$

where

$$D(x) = (x+\delta)^2 - \frac{4}{k+2}(x^2-1), \tag{6.32}$$

and $2A_\phi^M = \pm 1 - \cos\theta$ is a Dirac monopole connection where $\pm$ refers to the N(S) pole on the S^2. The ± 1 can be gauged away by *e.g.*, a shift of t to match the form given in section 6.1.

The B-field also simplifies in a similar way, and is of the general form $B_{t\phi} \sim \frac{f(x)}{D(x)}$, where $f(x)$ is a generically non-singular function of x and the other parameters (as confirmed for example by some numerical analysis to check the location of its divergences). I have not been able to find a pleasant closed form for $f(x)$, however. The problem of finding a short explicit expression is only a technical one, and due to the fact that the intermediate expressions are very long and complicated, and subject to the anomaly cancellation equations (6.13).

There are a total of four gauge fields in this model, A^1_μ, A^2_μ, B^1_μ and B^2_μ given by

$$A^m_\mu = \mathcal{G}^{mn}\mathcal{G}_{\mu n}, \qquad B^m_\mu = \mathcal{G}^{mn}B_{\mu n} = \mathcal{G}^{mn}(\mathcal{B}_{\mu n} + \mathcal{B}_{sn}A^s_\mu). \tag{6.33}$$

Again, there are quite remarkable simplifications leading to final expressions that are relatively nice and compact. The A's are found to be

$$A^1_t = \frac{(k-2)(x^2-1) + (k+2)(\delta + Q_A)(x+\delta)}{(k+2)D(x)}, \tag{6.34}$$

$$\begin{aligned} A^1_\phi = &\frac{-2A^M_\phi}{(k+2)D(x)}\Big[(k-2)^2 Q_B x^2 + (k^2-4)(\delta\lambda + 2\delta Q_B - Q_A\lambda)x \\ &+ (k+2)^2\delta(Q_B\delta - Q_A\lambda) + (k+2)(k-2)\lambda \\ &+ 8(P_A P_B Q_A - Q_B P_A^2 - Q_B)\Big], \end{aligned} \tag{6.35}$$

$$A^2_t = \frac{-\lambda(x^2-1) + P_A(x+\delta)}{D(x)}, \tag{6.36}$$

$$\begin{aligned} A^2_\phi = &\frac{-2A^M_\phi}{(k+2)(k-2)^2 D(x)}\Big[(k-2)^2[(k+2)\lambda^2 + (k-2)(P_B+1)]x^2 \\ &+ (k-2)^2(k+2)[-\lambda P_A + 2\delta(1+P_B)]x \\ &- \lambda^2(k+2)^2(k^2+4+8P_A^2) - \lambda(k+2)[(k-2)^2 - 16P_B]\delta P_A \\ &- 8((k+2)\delta^2 - k)P_B^2 + (k-2)^2(\delta^2(k+2)+4)P_B \\ &+ (k-2)[(k+2)^2\delta^2 - 8(P_A^2+1)]\Big]. \end{aligned} \tag{6.37}$$

The compact expressions for the B's are a little harder to get (for similar reasons to those given for the B-field), but I have at least a compact expression for one component:

$$\begin{aligned}B_t^1 = &\frac{1}{(k-2)(k+2)D}\Big[(k-2)^2x^2 \\ &+ [-(k-2)(k-4)Q_a + (k-2)(k+2)\delta - 2(k+2)(\lambda+Q_b)P_a]x \\ &+ 2(k-2)P_a^2 - 2(k+2)\delta Q_b P_a + k(k-2) - \delta^2(k^2-4) \\ &+ Q_a[2\lambda(k+2)P_a - \delta(k^2-4)]\Big].\end{aligned} \tag{6.38}$$

The gauge fields that appear in the heterotic nonlinear sigma model are not the fields written out above, but a linear combination of them. In the heterotic nonlinear sigma model, the gauge fields are in the left-moving sector, while the A's (B's) above correspond to symmetrised (antisymmetrised) combinations of left and right. The sigma model gauge field is therefore given as

$$\mathcal{A}_\mu^m = \frac{1}{2}(A_\mu^m + B_\mu^m). \tag{6.39}$$

The linear combination $\frac{1}{2}(A_\mu^m - B_\mu^m)$ corresponds to the spin connection, which appear in the right-moving sector of the sigma model. There is a subtlety in these identifications, because of the "shift" in the definition of the B vectors (6.33). However, this is irrelevant at leading order in k, and in this limit, the above identifications give the same result as reported in ref. [115].

As we saw in section 5.4.3, the exact dilaton is given by

$$e^{2\hat{\Phi}} = (\det A)^{-\frac{1}{2}}(\det \mathcal{G}_{mn})^{-\frac{1}{2}}, \tag{6.40}$$

where the determinants can now be written as follows. Define

$$p = k-2+2P_B, \quad q = (k+2)\delta + 2Q_B, \quad r = (k+2)\lambda + 2Q_B. \tag{6.41}$$

Then

$$\det A = \Delta(x) = \big[(k-2)px - (2P_A r - pq)\big]^2 + 4(r^2 - p^2), \tag{6.42}$$

$$\det \mathcal{G}_{mn} = 4(k+2)(k-2)^3\frac{D(x)}{\Delta(x)}. \tag{6.43}$$

The resulting exact dilaton is:

$$\hat{\Phi} - \hat{\Phi}_0 = -\frac{1}{4}\ln D(x), \tag{6.44}$$

where I have absorbed a non-essential constant into the definition of $\hat{\Phi}_0$.

With the above expressions for the exact spacetime fields, it is worth noting that they all diverge at $D(x) = 0$, and that this is the only place where they diverge. The coefficients $\mathcal{G}_{MN}$ and $\mathcal{B}_{MN}$, and the dilaton before re-fermionisation had divergences for $\Delta(x) = 0$, so what the re-fermionisation has done is to include the back-reaction in a way which moves these to the true singularity $D(x) = 0$. It is highly non-trivial that this happens so nicely for all the fields.

6.4.2 Properties of the Exact Metric

As already stated in the previous section, the final result for the exact spacetime metric is (after a trivial shift in t):

$$\begin{aligned} ds^2 &= (k-2)\left(\frac{dx^2}{x^2-1} + F(x)(dt - \lambda\cos\theta d\phi)^2 + d\theta^2 + \sin^2\theta d\phi^2\right) , \\ \text{where} \quad F(x) &= -\frac{x^2-1}{D(x)} = -\left(\frac{(x+\delta)^2}{x^2-1} - \frac{4}{k+2}\right)^{-1} . \end{aligned} \tag{6.45}$$

This is a pleasingly simple form to result from such an involved computation. In fact, its relation to the leading order result is reminiscent in form to the relation between the leading order and exact results for the black hole $SL(2,\mathbb{R})/U(1)$ model [104,105] of section 5.3.4.

It is interesting to sketch the behaviour of $G_{tt} = F(x)$, as it contains the answer to the original questions about the fate of the Taub and NUT regions of the spacetime once the contributions of the stringy physics are included. This result is plotted in figure 6.3, and it should be contrasted with figure 6.1.

Several remarks are in order. The first is that the Taub and NUT regions, although modified somewhat, survive to all orders. The second is that the local structure of the chronology horizons separating these regions is completely unaffected by the stringy corrections! $F(x)$ still vanishes at $x = \pm 1$ and furthermore for $x = 1 - \tau$ where τ is small, the metric of the (τ, ξ) space (the space over each point of the S^2) becomes:

$$ds^2 = (k-2)\left(-(2\tau)^{-1}d\tau^2 + \frac{2\tau}{(1+\delta)^2}d\xi^2\right) , \tag{6.46}$$

which is again of Misner form.

Notice that the singularity we observed in $F(x)$ (and the spacetime) has now split into two. Recalling the definition of $D(x)$ given in equation (6.32), we can

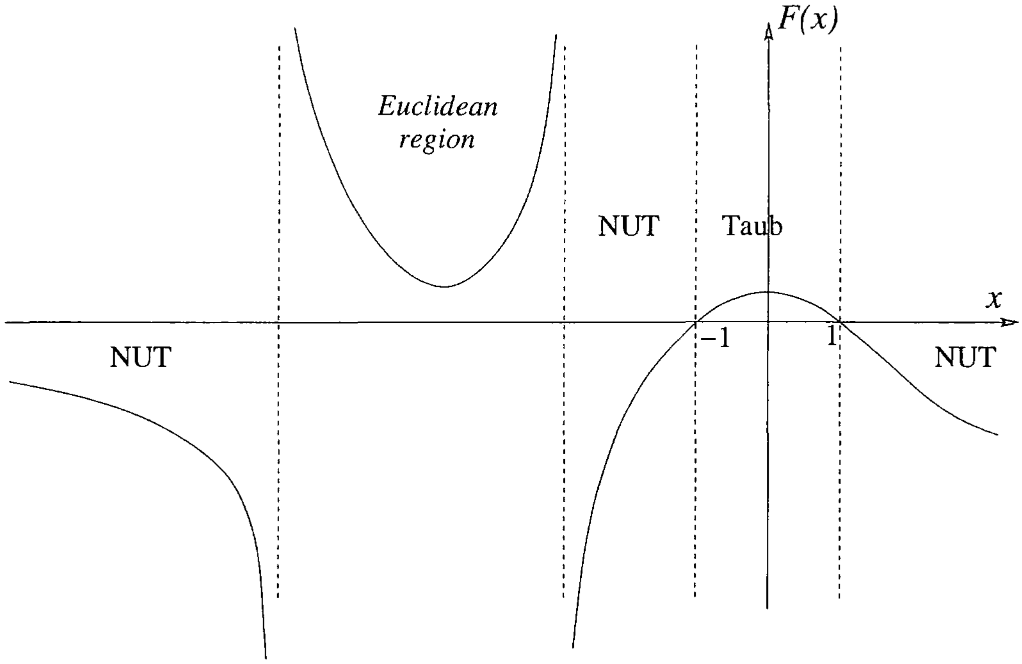


Figure 6.3: The various regions in the stringy Taub–NUT geometry for arbitrary k, with all $1/k$ corrections included. Compare to the leading order result in figure 6.1. Note that the singularity splits in order to incorporate a finite sized region of Euclidean signature in the second NUT region.

write the Ricci scalar as:

$$R = -\frac{1}{2(k-2)D^2}\left[2D(x^2-1)D'' - 3(x^2-1)\left(D'\right)^2 + \lambda^2(x^2-1)D + 6xDD'\right] , \tag{6.47}$$

(where a prime means d/dx). Note that at $x = \pm 1$, we have a finite, non-vanishing result for R for generic values of the parameters:

$$R = \pm 6\frac{\delta(k+2) \pm (k-2)}{(1\pm\delta)^2(k^2-4)} = \pm\frac{6}{(\delta \pm 1)k}\left(1 + \frac{2(\delta \mp 1)}{k} + \frac{4}{k^2} + \dots\right). \tag{6.48}$$

The curvature R diverges if and only if $D(x) = 0$. These singularities are located at:

$$x_\pm = \frac{-\delta \pm \sqrt{a^2 + a(\delta^2-1)}}{(1-a)} , \quad a = \frac{4}{k+2} , \tag{6.49}$$

and the region in between them has Euclidean signature. Such a region was noticed in ref. [179] in the context of the exact metric for the $SL(2,\mathbb{R})/U(1)$ coset giving the two dimensional black hole. This region remains entirely within the second NUT region, however, and never approaches the Misner horizons. Its size goes as $1/(k-2)$. The model only seems to make sense for $k > 2$, of course, and it is interesting to note that the limiting behaviour of this metric as $k \to 2^+$ is that the Euclidean region grows until it fills the entire left hand side of the sketch (see figure 6.4), with one singularity at $x = -(\delta^2+1)/(2\delta)$, and the other one moving off to $x \to -\infty$.

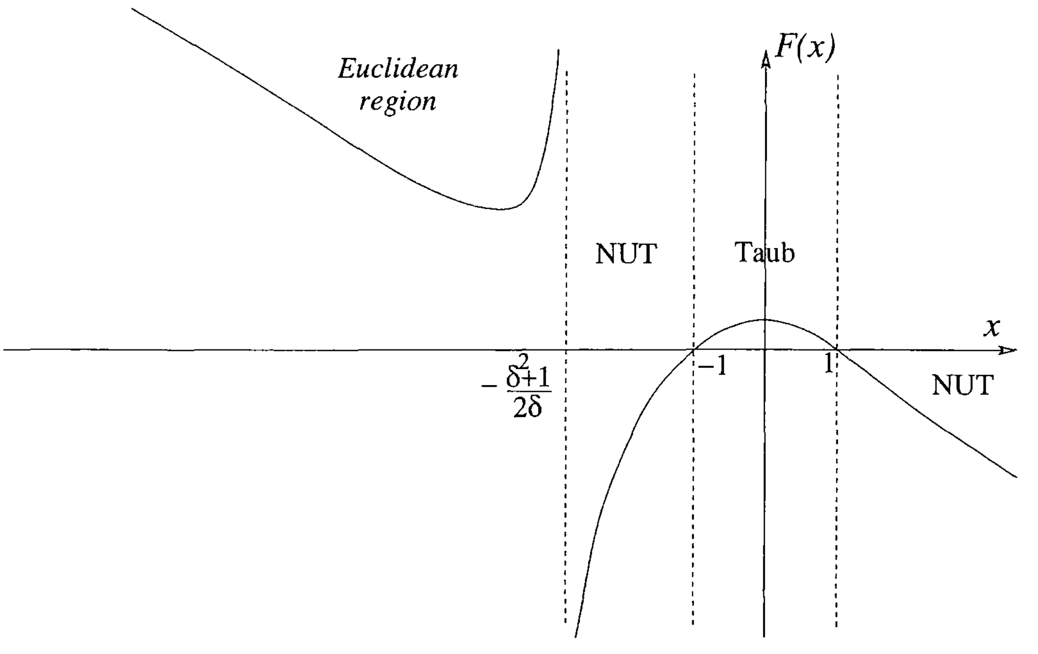


Figure 6.4: The various regions in the stringy Taub-NUT geometry for the smallest value of k possible. This is the "most stringy" geometry. Compare to the leading order result in figure 6.1 and the intermediate k result in figure 6.3. The Euclidean region has grown and occupied the entire region to the left, making the second nut region of finite extent.

6.5 Discussion

The goal in this chapter was to identify a stringy laboratory for the study of a number of issues of interest, which allows a controlled study of various physical phenomena. Closed timelike curves are very common in General Relativity, but the theory is silent about their physical role in a complete theory of gravity. They can appear after a cosmology passes through a certain type of spacelike "Big Crunch" singularity, and it is natural to wonder if the full theory somehow modifies the geometry in a way which obstructs this process of formation, realising the so-called chronology protection conjecture [74]. The model upon which a great deal of the study within General Relativity has been focused is the Taub-NUT spacetime (or local parts of it). Quite satisfyingly, this is precisely the model that we have studied here, furthering earlier work which showed how to embed it into string theory in a way which allows a complete definition in terms of conformal field theory.

What we have done is to go beyond the low energy truncation and compute the geometry exact to all orders in α', thereby including the effects of the entire string spectrum on the background. The embedding (into heterotic string theory) was chosen so as to permit such corrections to occur, at least in principle. Somewhat surprisingly (perhaps) we found that the key features of the Taub-NUT geometry persist to all orders. This includes the fact that the volume of the universe in the Taub cosmology vanishes as a circle shrinks to zero size, at the junction (described by Misner space) where the CTCs first appear. There is no disconnection of the Taub region from the NUT regions containing the CTCs, to all orders in α'. It is important to note that this is quite non-trivial (and so *not* to be compared to the flat space result) because the curvature does not vanish in the region of interest (see equation (6.48)). Note that the strength of the string coupling near the junctions is not particularly remarkable, and so an appeal to severe corrections purely due to string loops may not help modify the geometry further.

This work has therefore ruled out a large class of possible modification to the geometry which could have destroyed the chronology horizons and prevented the formation of the CTC regions (from the point of view of someone starting in the cosmological Taub region). As remarked upon in the introduction of the chapter, there is still the possibility that there is an instability of the *full* geometry to back-reaction by probe particles or strings. A large class of such effects are likely missed by the all orders computation of the metric. There are studies of Misner space in various dimensions (in its orbifold representations) that signal such an instability [138, 140, 141], and the fate of the chronology horizons embedded in this geometry should be examined in the light of those studies. The nature of the spacetime in which they are embedded is important, however, and so it seems that the relevant geometry to study such back-reaction effects is the fully corrected geometry I have derived here, since it takes into account the full α' effects.

Quantum effects may well be important even though the string coupling is not strong at the chronology horizons, and even if there are no (as we have seen here) modifications due to α' corrections. Radically new physics can happen if there are the right sort of special (for example, massless) states arising in the theory there together with (crucially) certain types of new physics. Strings wrapped on the t-circle are candidate such states. Following these states could shed new light on the validity of the geometry if they are accompanied by the appropriate physics, such as in the mechanism of ref. [70]. Such probe heterotic strings are hard to study in the sigma model approach, but it would be interesting to undergo such an investigation. The study of probes directly in the full conformal field theory (*i.e.*, without direct reference to the geometry) may well be the most efficient way to proceed.

Another (less often considered) possibility is that the result of this study is a sign that the theory is perfectly well-defined in this geometry. The conformal field theory is well-defined, and there are no obvious signs of a pathology. Perhaps string theory is able to make sense of all of the features of Taub-NUT. For example, the shrinking of the spatial circle away to zero size at the Big Bang or Big Crunch might not produce a pathology of the conformal field theory even though there might be massless states appearing from wrapped heterotic strings. They might simply be incorporated into the physics in a way that does not invalidate the geometry: The physics, as defined by the worldsheet model, would then carry on perfectly sensibly through that region. This would mean that there is another geometry that a dual heterotic string sees which is perfectly smooth through this region. It would be interesting to construct this geometry. Note that the right-handed worldsheet parity flip which generates a dual geometry is no longer achievable by axial-vector duality as in simpler cases such as the $SL(2,\mathbb{R})/U(1)$ black hole [105, 180]. It only works for $\delta = \pm 1$, $\lambda = 0$. Here, it is natural to explore whether $\delta \to -\delta$ combined with other actions might generate it, but a fibre-wise duality rather like that which relates [181, 182] an NS5-brane to an ALE space might be more appropriate.

In this scenario, if we accept that the conformal field theory is telling us that the stringy physics is well behaved as it goes through from the Taub region to the NUT region, we have to face the possibility that the CTCs contained in the NUT regions might well be acceptable, and part of the full physics as well.

While it is perhaps too early to conclude this with certainty, it is worth noting that most objections that are raised about physics with CTCs are usually ones based on paradoxes arrived at using macroscopic and manifestly classical reasoning, or reasoning based on our very limited understanding of quantum theory outside of situations where there is an asymptotic spacetime region to which we make reference. Some CTCs fall outside of those realms, opening up new possibilities. We must recall that time, just like space, is supposed to arrive in our physics as an approximate object, having a more fundamental quantum mechanical description in our theory of quantum gravity. The frequent occurrence of CTCs in theories of gravity might be a sign that (appropriately attended to) they are no more harmful than closed spatial circles. Rather than try to discard CTCs, we might also keep in mind the possibility that they could play a natural role in the full theory, when we properly include quantum mechanics. Here, we saw them remain naturally adjoined to a toy cosmology, surviving all α' corrections. This is just the sort of scenario where CTCs might play a role in nature: A natural way to render meaningless the usual questions about the lifetime of the universe prior to the "Big Bang" is to have the Big Bang phase adjoined to a region with CTCs[5]. This is an amusing alternative to the usual scenarios, and may be naturally realised within string theory, or its fully non-perturbative successor.

[5]Although it is in the very different context of eternal inflation, the role of CTCs in cosmology has been speculated about before [183].

Chapter 7

High energy corrections in a stringy Kerr-Taub-NUT solution

The low-energy limit of the stringy Taub-NUT spacetime discussed in the previous chapter is known [184] to be a special case of a larger family of solutions which as well as the NUT parameter λ also has an angular momentum parameter τ. This rotating solution is known as the Kerr-Taub-NUT spacetime. The throat + horizon region of the solution has furthermore been shown in ref. [185] to appear as the low-energy limit of an exact conformal field theory, defined as a heterotic coset model as described in chapter 5. Having thus another example of a gauged WZNW model, exact in α', it is interesting to carry out the same analysis as in chapter 6, and deduce the exact spacetime fields. The aim of this chapter is to do exactly this, and then to discuss some properties of the exact metric.

The construction of the heterotic coset model, and the computation of the spacetime fields are analogous to what I have done several times already, and I will only briefly summarise this in the following. The study of this chapter goes beyond that of chapter 6 in that we now give up spherical symmetry (for $\tau \neq 0$), and the main purpose of this chapter is to discuss some of the effects the rotation has on the spacetime.

7.1 Exact conformal field theory

The construction is based on the same coset as before, $SU(2) \times SL(2,\mathbb{R})/U(1)_A \times U(1)_B$, and fermions are included in exactly the same way as in chapter 6 to give a heterotic coset model [185]. Again, the model can be given a Lagrangian formulation as a gauged WZNW model. The parametrisations of the group elements is taken as before, given by equations (6.24, 6.25, 6.29). The new thing is the implementation of the gauge symmetry: In this chapter we shall impose the symmetry

$$\begin{aligned} g_1 &\longrightarrow e^{\epsilon_A \sigma_3/2} g_1 e^{(\delta\epsilon_A + \lambda\epsilon_B)\sigma_3/2}, \\ g_2 &\longrightarrow e^{i\tau\epsilon_A\sigma_3/2} g_2 e^{i\epsilon_B\sigma_3/2}, \end{aligned} \tag{7.1}$$

where $g_1 \in SL(2,\mathbb{R})$ and $g_2 \in SU(2)$. This should be compared to equation (6.12) in the stringy Taub-NUT case. It is apparent from the above gauging that the gauge group generators are still given by equations (6.17,6.28), except there is a new one: $t^{(2)}_{A,L} = i\tau\frac{\sigma_3}{2}$. For $\tau = 0$ we recover the model discussed in chapter 6, where the $SU(2)_L$ symmetry is unbroken, resulting in a spacetime metric with spherical symmetry. For $\tau \neq 0$, the gauging above breaks the $SU(2)_L$ symmetry, and therefore the metric will not have this symmetry anymore.

The above gauging is anomalous by itself, but by incuding fermions in their bosonised form, we can make the anomalies cancel by choosing the appropriate charges, as discussed in section 6.3.1. These classical anomaly cancellation equations are in this case:

$$\begin{aligned} k_1(\delta^2 - 1) - k_2\tau^2 &= 2(Q_A^2 + P_A^2 - \delta^2), \\ k_1\lambda^2 + k_2 &= 2(Q_B^2 + P_B^2 - (1+\lambda^2)), \\ k_1\delta\lambda &= 2(Q_A Q_B + P_A P_B - \lambda\delta), \end{aligned} \tag{7.2}$$

where, again, $k_2 = k_1 - 4$ to give the four-dimensional model a central charge $c = 6$. Parameters which satisfy these equations represent meaningful gauge-invariant theories. In the following I will write $k_1 = k$, $k_2 = k - 4$.

The gauge fixing is done by imposing

$$t_L = 0, \qquad \psi = 0. \tag{7.3}$$

This is slightly different from eq. (6.26) used in chapter 6. However, this difference corresponds to a gauge transformation on the resulting spacetime fields and is not important for our discussion.

As was the case in the Taub-NUT solution, the gauging (7.1) also induces a periodicity of the variable t_R, which becomes the time t with the gauge fixing (7.3). This can be deduced by studying the action of rotations on the resulting spacetime metric, but is also clear from the gauging (7.1): A $U(1)_B$ transformation with $\epsilon_B = 4\pi$ acts as the identity on the $SU(2)$ space, while in $SL(2,\mathbb{R})$ it translates $t_R \to t_R + 4\pi\lambda$. Hence gauging $U(1)_B$ identifies $t_R \sim t_R + 4\pi\lambda$ [185].

7.2 Low-energy limit

This heterotic coset model was constructed in ref. [185], and the spacetime fields in the large $k \sim \frac{1}{\alpha'}$ (low-energy) limit were found. The explicit expressions for the metric and the dilaton are[1]

$$
\begin{aligned}
ds^2 =& k\Big[\frac{dx^2}{x^2-1} - \frac{x^2-1}{(x+\delta-\lambda\tau\cos\theta)^2}(dt-\lambda\cos\theta d\phi)^2 \\
&+ d\theta^2 + \frac{\sin^2\theta}{(x+\delta-\lambda\tau\cos\theta)^2}(\tau dt-(x+\delta)d\phi)^2\Big], \\
e^{2(\hat{\Phi}-\hat{\Phi}_0)} =& (x+\delta-\lambda\tau\cos\theta)^{-\frac{1}{2}}.
\end{aligned}
\tag{7.4}
$$

This solution has the same Killing horizons at $x = \pm 1$ as the non-rotating model, and a curvature singularity for $x = -\delta + \lambda\tau\cos\theta = 0$. There is also an ergosphere (both in the positive and negative x domains) given by $1 < x^2 < 1 + \tau^2\sin^2\theta$ outside the horizon where the rotational frame dragging makes it impossible for any particle to remain stationary. I will come back to these various regions shortly, when discussing the exact metric. Except for the ergosphere, the overall structure of this spacetime is similar to the stringy Taub-NUT in the low-energy limit, illustrated in figure 6.1.

As in the stringy Taub-NUT case, there are closed timelike curves (CTCs) in the NUT regions $x^2 > 1 + \tau^2\sin^2\theta$, where t is timelike and periodic. The region $-1 < x < 1$ is the cosmological Taub patch, where t is spacelike, and x is timelike. In this region, the singularities at $x = -1$ and $x = 1$ correspond to a Big Bang, and a Big Crunch respectively, just as in the previous chapter.

Another crucial observation made about the stringy Taub-NUT in the throat + horizon region we are investigating, was that it is Misner-like in the neighbourhood of the horizons, see eq. (6.9). This is still true with rotation, and therefore the semi-classical study of ref. [125] is still relevant. –Recall that this study shows that the vacuum stress-energy tensor for a conformally coupled scalar field in the background diverges at the horizons, indicating an infinite back-reaction that is often believed to be such that CTCs are avoided in the exact geometry.

[1] I have written the extended version where $\cosh\sigma \to x$ and x can take any real value. Note also that $\hat{\Phi}(\text{here}) = \frac{1}{4}\hat{\Phi}(\text{there})$.

7.3 Exact metric and dilaton

The techniques used for deriving the exact spacetime fields corresponding to a heterotic coset model can be summarised as follows.

1. Find a Lagrangian formulation of the model in terms of gauged WZNW models, where the fermions are included in their bosonised form.
2. Change to variables where the Lagrangian is a sum of ungauged WZNW Lagrangians, and remember to take into account the Jacobian.
3. Deduce the effective action, which is done merely by shifting the level constants.
4. Change variables back to the original ones. (Note that there is no Jacobian this time, as the fields are classical.)
5. Integrate out the gauge fields.
6. Prepare the action for re-fermionisation.
7. Read off the exact spacetime fields.

This has been discussed and demonstrated in detail earlier on, so I just jump directly to the result for the present model. The exact metric is found to be

$$\begin{aligned}
ds^2 =&(k-2)\Big[\frac{dx^2}{x^2-1}-\frac{E(x,\theta)}{D(x,\theta)}\Big(dt+\frac{\Lambda(x,\theta)}{E(x,\theta)}d\phi\Big)^2+d\theta^2+\frac{x^2-1}{E(x,\theta)}\sin^2\theta d\phi^2\Big],\\
&E(x,\theta)=x^2-1-\tau^2\sin^2\theta,\\
&D(x,\theta)=(x+\delta-\lambda\tau\cos\theta)^2-\frac{4}{k+2}(x^2-1-\tau^2\sin^2\theta),\\
&\Lambda(x,\theta)=-\lambda\cos\theta(x^2-1)+\tau\sin^2\theta(x+\delta).
\end{aligned} \tag{7.5}$$

As expected, for $\tau=0$ this reduces to the non-rotating solution (6.45), and for $k\to\infty$ we recover the low-energy metric (7.4).

The general expression for the dilaton is

$$e^{2\Phi}=(\Delta)^{-\frac{1}{2}}(\det\mathcal{G}_{mn})^{-\frac{1}{2}}, \tag{7.6}$$

where one part is from integrating out the gauge fields, and the other part is from re-fermionisation. These are given by

$$\begin{aligned}
\Delta(x,\theta)&=\Big[(k-2)px+pq-(2P_A+(k-2)\tau\cos\theta)r\Big]^2+4(1-\tau^2)\Big[r^2-p^2\Big],\\
\det\mathcal{G}_{mn}&=4(k+2)(k-2)^3\frac{D(x,\theta)}{\Delta(x,\theta)},
\end{aligned} \tag{7.7}$$

where p, q and r have been defined previously in eq. (6.41). This gives the exact dilaton

$$\hat{\Phi} - \hat{\Phi}_0 = -\frac{1}{4} \ln D(x, \theta). \tag{7.8}$$

7.3.1 Properties of metric

The metric (7.5) has two Killing vectors, $\xi = \frac{\partial}{\partial t}$ and $\psi = \frac{\partial}{\partial \phi}$ representing time translation symmetry, and axial symmetry. The Killing horizon is defined as the surface where a linear combination of the Killing vectors becomes null [186]. This happens for the Killing vector

$$\chi = \frac{\partial}{\partial t} + \Omega_H \frac{\partial}{\partial \phi}; \qquad \Omega_H = \frac{\tau}{\delta \pm 1}, \tag{7.9}$$

at $x = \pm 1$, since the length there is

$$\chi^2 = G_{tt} + 2G_{t\phi}\Omega_H + G_{\phi\phi}\Omega_H^2 = \frac{k-2}{D}(\tau - (\delta \pm 1)\Omega_H)^2 = 0. \tag{7.10}$$

So there are Killing horizons at $x = -1$ and at $x = 1$. The quantity Ω_H is interpreted as the angular velocity at the horizon and is proportional to τ as anticipated. Note that the horizon is independent of the value of all the parameters, so it is the same as in the low-energy limit, and also the same as in the non-rotating case. Note also that the angular velocity of the horizon, Ω_H is independent of k.

The metric component G_{tt} becomes zero when $E(x, \theta) = 0$ *i.e.*, when

$$x^2 = 1 + \tau^2 \sin^2\theta \quad \in \ [1, 1 + \tau^2]. \tag{7.11}$$

This surface, which I shall call the ergosurface, lies outside the horizon (see figure 7.1). Between the horizon and this ergosurface is the ergosphere, a region where no stationary particles can exist. To see that the ergosphere really is a region where particles cannot be stationary, consider the following. Assume that a particle follows a trajectory with tangent vector u, which has to be timelike *i.e.*, $u^2 = 0$. In the stationary case, the only motion is in the time direction, so the tangent vector is $u = u^t \frac{\partial}{\partial t}$, where $u^t = \frac{dt}{ds}$, and s is proper time along the curve. But this gives $u^2 = G_{tt}(u^t)^2 > 0$ in the ergosphere (where $G_{tt} > 0$), and so the assumptions are inconsistent: No stationary motion is possible in the ergosphere. What happens instead is that particles are affected by the *rotational frame dragging* and inevitably follow the rotation of the black hole (as observed from infinity).

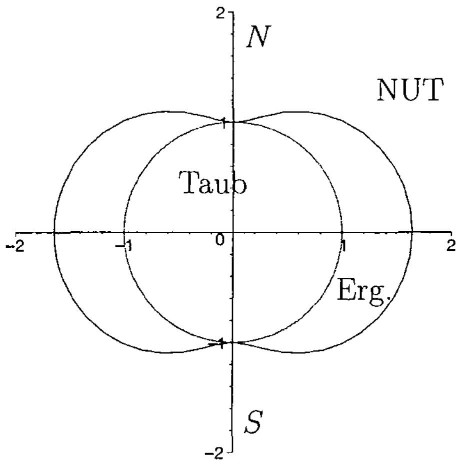


Figure 7.1: Polar plot of the locus of the horizon (inner circle) and ergosurface (outer deformed circle) of stringy Kerr-Taub-NUT spacetime. The radial direction is $\sqrt{|x|}$, and the angle θ runs from 0 at the north pole (N) to π at the south pole (S). (This plot is for $\tau = 2.5$.)

Curvature singularities

The metric (7.5) is ill defined for $D(x, \theta) = 0$, which is also a true curvature singularity. This can be verified by computing the curvature. It is then seen that both the Ricci scalar and the Kretschmann scalar ($R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$) behave like $\sim \frac{1}{D(x,\theta)^2}$, and so indeed $D = 0$ represents a curvature singularity. (The same argument would also show that $x^2 - 1 = 0$ or $E = 0$ are *not* curvature singularities.)

First of all, notice that $D = 0$ only has solutions if

$$x^2 - 1 - \tau^2 \sin^2\theta = E(x, \theta) > 0, \tag{7.12}$$

that is, singularities are only found outside the ergosphere. (Inside the ergosurface, D is always positive.) The equation $D(x, \theta) = 0$ solved for x gives

$$\begin{aligned} x = x_\pm(\theta) &= -\frac{1}{k-2}(\beta \mp \sqrt{\alpha^2}), \\ \alpha^2 &= 4(k+2)(\delta - \lambda\tau\cos\theta)^2 - 4(k-2)(1 + \tau^2 \sin^2\theta), \\ \beta &= (k+2)(\delta - \lambda\tau\cos\theta). \end{aligned} \tag{7.13}$$

For given values of the parameters and for θ, the solutions $x_\pm$ have the same sign (which is the same sign as $-\beta$), as can easily be seen from the following:

$$\beta^2 - \alpha^2 = (k+2)(k-2)(\delta - \lambda\tau\cos\theta)^2 + 4(k-2)(1 + \sin^2\theta) \geq 0. \tag{7.14}$$

Hence, $|\beta| \geq |\alpha|$.

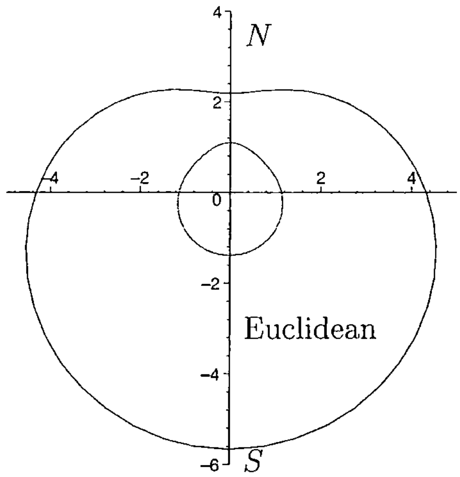


Figure 7.2: Polar plots of the locus of the singularities for the small τ case. The radial direction is $\sqrt{|x|}$, and the angle θ is zero at the north pole (N) and π at the south pole (S). There is symmetry in the ϕ direction. (This plot is for $k = 3, \delta = 2, \lambda = 14, \tau = 0.1$.)

The singularities $x_\pm(\theta)$ can be divided into two classes, depending on the values of the parameters k, λ, δ, τ. The first class is the *small rotation case*, which has relatively small τ (I will come back to what this means). In this case α^2 is always positive, and $x_\pm$ exist for all θ, and are both negative. This situation is illustrated in figure 7.2. The spacetime structure in this case is a smooth deformation of the non-rotating stringy Taub-NUT spacetime of the previous chapter, illustrated in figure 6.3. The singularities appear in the negative x region, and enclose a region of Euclidean signature. Seen from the positive $x > 1$ NUT region, and the from the Taub region $-1 < x < 1$ this is a singularity hidden behind a horizon, while from the negative $x < -1$ NUT regions they appear as naked singularities.

The second class is the *large rotation case*, where τ in some sense is large. In this case, α^2 becomes negative for some values of θ. Hence, for these angles, there are no divergences. This is illustrated in figure 7.3. What happens is that the two surfaces $x \pm (\theta)$ connect and form a "bubble" outside the ergosphere. One such bubble is centered at the south pole ($\theta = \pi$) and appear in the negative x region. This also makes the two NUT regions in the negative x domain merge together into one connected region. Another bubble may or may not appear at the north pole ($\theta = 0$) for positive x. This is rather different from the non-rotating case, and quite exotic behaviour. The bubbles still enclose regions of Euclidean signature, but since they appear in both the positive and the negative x region, all the NUT regions are plagued by naked singularities.

The meaning of small and large rotation refers to which of the above situations it leads to. Assuming k, δ, λ are given, there is a critical value for τ above which α^2

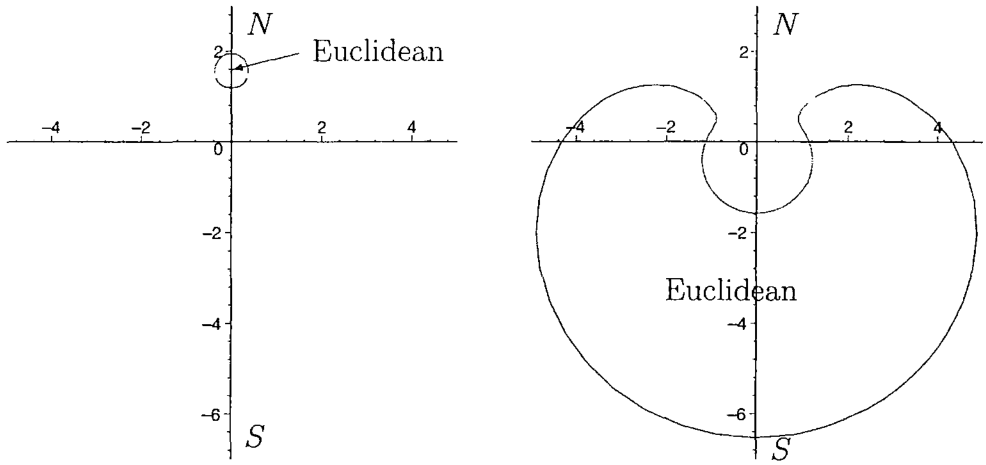


Figure 7.3: Polar plots of the locus of the singularities for the large τ case. (This plot is for $k = 3, \delta = 2, \lambda = 14, \tau = 0.18$.)

in equation (7.13) is *not* positive definite. Let me call this critical value $\tau_c(k, \delta, \lambda)$. Then "small rotation" means values $\tau < \tau_c$, and "large rotation" means values $\tau > \tau_c$.

7.4 Discussion

The model discussed in this section is a generalisation of the stringy Taub-NUT spacetime of the previous chapter. The rotational symmetry is broken in the general case. We saw in chapter 6 that the α' corrections do not modify the spacetime significantly with regards to the CTCs in the non-rotating case, and this result persists in the rotating case. So all the comments made there carry on to the more general model of this chapter. This is no surprise, but nonetheless a valuable observation in that it shows us that the results of chapter 6 are not simply a coincidence happening only for that particular spacetime. Noting the miraculous cancellation that gave the simple form for the exact metric, we could have been tempted to believe there was something very special happening in that case. Now, as we see the same happening again, an interpretation of it as a mere coincidence seems even more unlikely. A more reasonable interpretation of the mild α' corrections near the horizons seems to be that string theory really does not rule out the possibility of CTCs. This view has already been discussed in section 6.5.

The singularities in the present model differ from what we saw in the non-rotating case, and this deserves a comment. First of all, it is important to keep in mind that the dilaton blows up at the singularity, so the string coupling g_s is in no sense small. Hence, g_s corrections may completely alter the geometry at the "would-be-singularities" where $D(x, \theta) = 0$. How to compute these corrections, however, is beyond reach with our technology at present. So the exotic singular structure of the metric (7.5) might only be an artefact of working in the classical limit (which is a good approximation only if $g_s \to 0$).

If we ignore this for a moment, we have spacetimes containing problematic naked singularities in the NUT regions. If the rotation is mall, these appear only for negative x, and so we could still make sense of the positive x NUT region since it would be protected from the singularities by the horizons at $x = \pm 1$. In this case the Taub region has a natural extension past $x = 1$ into the region $x > 1$, giving a cosmology with a post Big Crunch scenario. But for large rotation, the singularities appear both in the positive and negative x NUT regions, and any sensible extension of the Taub region seems impossible.

Chapter 8

Conclusion

In this thesis I have studied a maximally supersymmetric plane wave with RR flux and a stringy Taub-NUT spacetime as examples of exact solutions of string theory. These belong to the two main classes of exact string theory solutions, which are the plane waves and the gauged Wess-Zumino-Novikov-Witten (WZNW) models. The prime goal of the thesis has been to investigate various properties of these solutions to all orders in the inverse string tension α', thereby taking into account stringy high-energy effects. This is interesting already without further motivation because of the scarcity of exact solutions. In the present situation, any calculation beyond the leading α' limit that *can* be done, seems worthwhile to do. String theory is yet not a fully developed theory, and one of the obstacles to a better understanding of it is the fact that we most often are restricted to particular limits of the full theory when we try to do explicit calculations. For example is the low-energy ($\alpha' \to 0$) supergravity limit much more tractable than the full string theory. Plane waves and gauged WZNW models provide valuable examples where we can in fact surpass the low-energy limit.

In the case of the plane wave solution, I investigated how the stringy halo of D-branes is modified as compared to the halo in flat space. I also commented on the connection between the Hagedorn temperature and the radius of self-duality under a T-duality transformation, and how an understanding of this in the plane wave case is obstructed by the poor understanding of T-duality in null directions.

This observation motivated the study of chapter 4, where I investigated T-dualities which are spacelike, null or timelike depending on the value of a particular parameter. This was done in a spinning D-brane solution of supergravity, which contains closed timelike curves (CTCs).

The wish to understand the role of CTCs in string theory was in turn part of the motivation for studying the stringy Taub-NUT solution. Being an exact solution, it provides a perfect laboratory for a controlled investigation of CTCs beyond the supergravity limit. Also other aspects of the Taub-NUT solution, such as the analytic extension of the metric and the interpretation of it as a cosmological solution, motivated the investigation and were discussed.

The investigation of D-branes in the plane wave background in chapters 2 and 3 showed that the stringy halo is the same as in flat space for Lorentzian D-branes, while it has a non-trivial modification in the case of Euclidean D-branes. The difference between the two cases has to do with the choice of light-cone gauge. The gauge choice has to be consistent with the assumption about the signature of the worldvolume, and has to be done differently for the two classes of D-branes. For Lorentzian D-branes the result arises from the fact that the divergence defining the halo comes from a domain where the mass parameter goes to zero, thus giving the same as in flat space. The modification that occurs for Euclidean D-branes is harder to understand and to interpret. A complicating factor is that the calculation relies on a Wick rotation that makes the metric complex. This is a subtlety that I have not discussed in any detail, but which might be important for the interpretation of the result. Despite this, it seems clear that there *is* a modification from the flat space result. The stringy halo gives the minimum separation for which tachyon condensation occurs, which is relevant information when trying to classify D-branes via K-theory. The results reported here suggest that such a classification in the plane wave case should, at least in principle, be doable for Lorentzian D-branes, while more complicated for Euclidean D-branes.

The investigation of T-duality in chapter 4 showed that the divergence associated with null T-duality is invisible from the point of view of probe strings and probe D-branes. This is a sign that the divergence might be an artefact of the supergravity solution (and the supergravity T-duality rules), and represents no problem in the full string theory once the α' corrections have been taken into account. Additionally, the CTCs appearing in these solutions were found to survive the T-dualities and also, *not* to be geodesics – a result in line with previous studies of CTCs in supergravity solutions.

The gauged WZNW models of chapters 5, 6 and 7 were investigated to all orders in α', and the exact spacetime fields were deduced for a stringy Taub-NUT solution, as well as for a rotating generalisation of it. I found that the exact fields are only mildly modified from the low-energy result near the horizon, and therefore that the issue of CTCs remains the same. A scenario where a (long-lived) observer starts at a Big Bang, goes through a Taub phase of expansion and then contraction *through* a Big Crunch into a NUT region on the "other side" which has CTCs, still seems a perfectly valid one after the inclusion of all α' corrections. The high-energy α' corrections fail to produce modifications that forbid the formation of CTCs. We are therefore led to speculate whether other stringy effects such as loop corrections can do this, or whether string theory simply has no problem with CTCs. An argument that string loops would not be important in this respect is that the string coupling is set by the dilaton, which in turn is not particularly large at the horizon connecting the Taub and NUT regions. So the calculation of the exact spacetime fields in this thesis gives some evidence that CTCs are in fact present in full string theory. And since the string theory has a perfectly well defined formulation as a conformal field theory, that would mean that these CTCs represent no problem in string theory and might just be a natural ingredient. This is an exciting possibility which immediately leads to the question of what their exact role in string theory really is.

Another interesting observation based on these calculations of the exact geometry concerns the curvature singularity. In some sense the singularity "blows up" when we go beyond the leading α' result, and enclose a new region of Euclidean signature. In the rotating case this is even more exotic, since these Euclidean regions might become "bubbles" outside the ergospheres, appearing in both NUT regions. We should be very careful with the interpretation of this, however. The dilaton, and therefore the string coupling, blows up at the horizon, and so the string loop corrections cannot be neglected.

"This is the end, beautiful friend, the end."[1]

[1]Lyrics by The Doors.

Appendix A

Definition of functions

A.1 Generalised Jacobi functions

Let

$$q = e^{-2\pi t}, \qquad \tilde{q} = e^{-2\pi s}, \qquad m = \tilde{m}s, \qquad t = \frac{1}{s}. \tag{A.1}$$

The Jacobi f-functions are defined as

$$\begin{aligned}
f_1(q) &= q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n), & f_2(q) &= \sqrt{2}q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1+q^n), \\
f_3(q) &= q^{-\frac{1}{48}} \prod_{n=1}^{\infty}(1+q^{n-\frac{1}{2}}), & f_4(q) &= q^{-\frac{1}{48}} \prod_{n=1}^{\infty}(1-q^{n-\frac{1}{2}}).
\end{aligned} \tag{A.2}$$

It is convenient also to define what we may refer to as deformed Jacobi f-functions: [41]:

$$\begin{aligned}
f_1^{(m)}(q) &= q^{-\Delta_m}(1-q^m)^{\frac{1}{2}} \prod_{n=1}^{\infty}(1-q^{|\omega_n|}), & f_2^{(m)}(q) &= q^{-\Delta_m}(1+q^m)^{\frac{1}{2}} \prod_{n=1}^{\infty}(1+q^{|\omega_n|}), \\
f_3^{(m)}(q) &= q^{-\Delta'_m} \prod_{n=1}^{\infty}(1+q^{|\omega_{n-1/2}|}), & f_4^{(m)}(q) &= q^{-\Delta'_m} \prod_{n=1}^{\infty}(1-q^{|\omega_{n-1/2}|}),
\end{aligned} \tag{A.3}$$

where $\omega_n = \mathrm{sign}(n)\sqrt{m^2+n^2}$, and Δ_m and Δ'_m are defined by

$$\begin{aligned}
\Delta_m &= -\frac{1}{(2\pi)^2} \sum_{p=1}^{\infty} \int_0^{\infty} ds\; e^{-p^2 s} e^{-\frac{\pi^2 m^2}{s^2}}, \\
\Delta'_m &= -\frac{1}{(2\pi)^2} \sum_{p=1}^{\infty} (-1)^p \int_0^{\infty} ds\; e^{-p^2 s} e^{-\frac{\pi^2 m^2}{s^2}}.
\end{aligned} \tag{A.4}$$

These satisfy

$$f_1^{(\tilde{m})}(\tilde{q}) = f_1^{(m)}(q), \qquad f_2^{(\tilde{m})}(\tilde{q}) = f_4^{(m)}(q), \qquad f_3^{(\tilde{m})}(\tilde{q}) = f_3^{(m)}(q). \tag{A.5}$$

The quantities Δ_m and Δ'_m are the Casimir engergies of a single (two-dimensionsal) boson of mass m on a cylindrical world-sheet with periodic and anti-periodic boundary conditions respectively. For $m = 0$ the $f^{(m)}$-functions reduce to the original f-functions (A.2), and $\Delta_0 = -\frac{1}{24}$, $\Delta'_0 = \frac{1}{48}$, which are the flat space values.

There are also other functions which appear in various cylinder diagrams [38]:

$$\begin{aligned}
g_1^{(m)}(q) &= 4\pi i m q^{-2\Delta_m} q^{m/2} \prod_{n=1}^{\infty}\Big(1 - \frac{\omega_n + m}{\omega_n - m} q^{\omega_n}\Big)\Big(1 - \frac{\omega_n - m}{\omega_n + m} q^{\omega_n}\Big),\\
g_2^{(m)}(q) &= 4\pi m q^{-2\Delta_m} q^{m/2} \prod_{n=1}^{\infty}\Big(1 + \frac{\omega_n + m}{\omega_n - m} q^{\omega_n}\Big)\Big(1 + \frac{\omega_n - m}{\omega_n + m} q^{\omega_n}\Big),\\
g_3^{(m)}(q) &= 2q^{-2\Delta'_m} \prod_{n=1}^{\infty}\Big(1 + \frac{\omega_{n-1/2} + m}{\omega_{n-1/2} - m} q^{\omega_{n-1/2}}\Big)\Big(1 + \frac{\omega_{n-1/2} - m}{\omega_{n-1/2} + m} q^{\omega_{n-1/2}}\Big),\\
g_4^{(m)}(q) &= 2q^{-2\Delta'_m} \prod_{n=1}^{\infty}\Big(1 - \frac{\omega_{n-1/2} + m}{\omega_{n-1/2} - m} q^{\omega_{n-1/2}}\Big)\Big(1 - \frac{\omega_{n-1/2} - m}{\omega_{n-1/2} + m} q^{\omega_{n-1/2}}\Big),
\end{aligned} \tag{A.6}$$

$$\begin{aligned}
\hat{g}_1^{(m)}(q) &= q^{-\tilde{\Delta}_m} \prod_{l\in\mathcal{M}_+} (1 - q^{\omega_l})^{\frac{1}{2}} \prod_{l\in\mathcal{M}_-} (1 - q^{\omega_l})^{\frac{1}{2}},\\
\hat{g}_2^{(m)}(q) &= q^{-\tilde{\Delta}_m} \prod_{l\in\mathcal{M}_+} (1 + q^{\omega_l})^{\frac{1}{2}} \prod_{l\in\mathcal{M}_-} (1 + q^{\omega_l})^{\frac{1}{2}},\\
\hat{g}_3^{(m)}(q) &= q^{-\hat{\Delta}_m} \prod_{l\in\mathcal{P}_+} (1 + q^{\omega_l})^{\frac{1}{2}} \prod_{l\in\mathcal{P}_-} (1 + q^{\omega_l})^{\frac{1}{2}},\\
\hat{g}_4^{(m)}(q) &= q^{-\hat{\Delta}_m} \prod_{l\in\mathcal{P}_+} (1 - q^{\omega_l})^{\frac{1}{2}} \prod_{l\in\mathcal{P}_-} (1 - q^{\omega_l})^{\frac{1}{2}},
\end{aligned} \tag{A.7}$$

where

$$\begin{aligned}
\tilde{\Delta}_m &= -\frac{1}{(2\pi)^2} \sum_{p=1}^{\infty}\sum_{r=0}^{\infty} c_r^p m \frac{\partial^r}{(\partial m^2)^r} \frac{1}{m} \int_0^{\infty} ds \Big(\frac{-2}{\pi^2}\Big)^r e^{-p^2 s - \frac{\pi^2 m^2}{s}},\\
\hat{\Delta}_m &= -\frac{1}{(2\pi)^2} \sum_{p=1}^{\infty}(-1)^p \sum_{r=0}^{\infty} c_r^p m \frac{\partial^r}{(\partial m^2)^r} \frac{1}{m} \int_0^{\infty} ds \Big(\frac{-2}{\pi^2}\Big)^r e^{-p^2 s - \frac{\pi^2 m^2}{s}},
\end{aligned} \tag{A.8}$$

and c_r^p are Taylor coefficients of the functions

$$(\frac{x+1}{x-1})^p + (\frac{x-1}{x+1})^p = \sum_{r=0}^{\infty} c_r^p x^{2r}. \tag{A.9}$$

The sets $\mathcal{P}_\pm$ and $\mathcal{M}_\pm$ are defined by *positive* numbers n which satisfy

$$\begin{aligned} n \in \mathcal{P}_+ : \quad \frac{n+i\tilde{m}}{n-i\tilde{m}} &= -e^{i2\pi n}, \qquad n \in \mathcal{M}_+ : \quad \frac{n+i\tilde{m}}{n-i\tilde{m}} = e^{i2\pi n}, \\ n \in \mathcal{P}_- : \quad \frac{n-i\tilde{m}}{n+i\tilde{m}} &= -e^{i2\pi n}. \qquad n \in \mathcal{M}_- : \quad \frac{n-i\tilde{m}}{n+i\tilde{m}} = e^{i2\pi n}. \end{aligned} \tag{A.10}$$

The above functions satisfy

$$\begin{aligned} g_1^{(\tilde{m})}(\tilde{q}) &= \hat{g}_1^{(m)}(q), \qquad g_2^{(\tilde{m})}(\tilde{q}) = \hat{g}_4^{(m)}(q), \\ g_3^{(\tilde{m})}(\tilde{q}) &= \hat{g}_3^{(m)}(q). \qquad g_4^{(\tilde{m})}(\tilde{q}) = \hat{g}_2^{(m)}(q). \end{aligned} \tag{A.11}$$

A.2 Deformed theta functions

The deformed theta functions are [187]:

$$Z_{\alpha,\beta}^{(m)}(\tau_1,\tau_2) = e^{4\pi\tau_2\Delta_b^{(m)}} \prod_{n=-\infty}^{\infty} \left(1 - e^{2\pi\tau_2\omega_{n+b} + 2\pi i\tau_1(n+b) + 2\pi i a}\right)\left(1 - e^{-2\pi\tau_2\omega_{n-b} + 2\pi i\tau_1(n-b) - 2\pi i a}\right). \tag{A.12}$$

The quantity

$$\Delta_b^{(m)} = -\frac{2}{(2\pi)^2}\sum_{p=1}^{\infty}\int_0^\infty ds e^{-p^2 s - \frac{\pi^2 m^2}{s}} \cos(2\pi b p) \tag{A.13}$$

is the Casimir energy of a 2D complex scalar boson ϕ of mass m with twisted boundary conditions

$$\phi(\tau, \sigma+\pi) = e^{2\pi i b}\phi(\tau,\sigma). \tag{A.14}$$

By comparing to the Casimir energies defined in eq. (A.4) we see that

$$2\Delta_m = \Delta_0^{(m)}, \qquad 2\Delta'_m = \Delta_{\frac{1}{2}}^{(m)}. \tag{A.15}$$

The above deformed theta functions are related to the generalised Jacobi f-functions (A.3) by (using $q = e^{-2\pi\tau_2}$):

$$\begin{aligned} f_1^{(m)}(q) &= \left[Z_{0,0}^{(m)}(\tau_1=0)\right]^{\frac{1}{4}}, \qquad f_2^{(m)}(q) = \left[Z_{\frac{1}{2},0}^{(m)}(\tau_1=0)\right]^{\frac{1}{4}}, \\ f_3^{(m)}(q) &= \left[Z_{\frac{1}{2},\frac{1}{2}}^{(m)}(\tau_1=0)\right]^{\frac{1}{4}}, \qquad f_4^{(m)}(q) = \left[Z_{0,\frac{1}{2}}^{(m)}(\tau_1=0)\right]^{\frac{1}{4}}. \end{aligned} \tag{A.16}$$

Appendix B

Summary of thesis for laymen

Physics

The ultimate goal of science is to understand all aspects of nature. To understand human nature is the prime interest of psychology. But if you want to understand particular aspects of humans at a deeper, more basic level, you are led to the discipline of biology, whose prime interest is the understanding of life. If you are still not satisfied, but want to understand the underlying principles of life, you have to study the various chemical processes and the properties of molecules and atoms, which brings you to the domain of chemistry. If you carry on with questions about what atoms are, what they are made of, and how they behave, you enter the field of physics, whose interest is the understanding of the basic laws of particles and their interactions. The formulation of these laws requires mathematics, which is the "mother language" of physics. The laws of physics are expressed as mathematical equations, and the study of these equations is the interest of *theoretical physics*.

Ideally, the study of the equations can be done in dialogue with experiments – the theoretician gives the experimentalist predictions, and the experimentalist gives the theoretician results. And they had better agree! There are circumstances, however, where experimental results simply are beyond reach. Then the theoretician is on his or her own, and guided only by intuition, and by mathematical consistency. In some sense this is then less physics and more mathematics, and the term *mathematical physics* might be more appropriate.

Last century saw two major advances in our understanding of nature at the level of fundamental physics. One was the General Theory of Relativity, which is a theory of gravitation and the universe on a large scale. The other was Quantum Field Theory, or the Standard Model, which is a theory about the elementary particles. These two theories have both been verified to impressive accuracy, and there is little doubt that they are correct descriptions of nature.

But there is a flaw: They are inconsistent! Usually this is not a problem, since the disagreement is only relevant in very exotic circumstances (*extremely* high energy density) that are rarely encountered in nature. But it is a fundamental inconsistency that *is* important for example to understand black holes, or Big Bang – the beginning of the universe. It is not experiments or experience that is telling us something is missing with our theories, but the theories themselves are saying this.

The search for a theory which resolves this inconsistency – a Quantum Gravity theory – has been the main long-term goal in theoretical/mathematical physics over the last few decades. This is where string theory enters the scene.

String theory

The basic assumption of string theory is that the fundamental building blocks are not point particles (like quarks and electrons), but tiny vibrating strings with a length of order 10^{-34} m. (To get an idea of the scale, imagine a string magnified up to 1 mm. Then an atom would appear to be as big as our entire galaxy. Alternatively, if the string were as big as an atom, an atom would be as big as the solar system.) Different vibrational modes of the string correspond to different particles, like different vibrations of a guitar string give rise to different tones. This is the basic idea, but I should emphasise straight away that it is far from clear how precisely to identify the particles we observe in nature (electrons, quarks etc.) within string theory. Nevertheless, the very good thing about string theory is that it naturally contains both gravity and elementary particle physics, and is a *consistent* unified theory of Quantum Gravity. Hurrah! The only problem is: How do we get the type of gravity and the type of elementary particles we see in nature?

String theory has an extremely rich structure, and in many ways it might be better to think of it as a framework for theoretical physics rather than a definite model as such. Still, it is the hope that string theory will one day be understood well enough to make it possible to explain the origin of both gravity and the Standard Model. At present, most of the work done in string theory is aimed at understanding the theory itself (leaving for later the questions of how it relates to the real world).

Although it has not made direct connection with the observable world, the study of string theory has led to the discovery of many connections within theoretical physics, and even mathematics, and has provided a better understanding of concepts already well established in other areas of theoretical physics. For example, it predicts an equivalence between certain theories of elementary particles (quantum field theory) and certain theories of gravity that has opened up new windows into the understanding of each of them. This is a valuable result regardless of how or whether at all string theory connects to the real world itself.

The thesis

String theory, like other models in theoretical physics, comes with fundamental field equations. These are differential equations which generally have many solutions, but are usually hard to solve. Solving differential equations also requires knowledge of (and is highly dependent upon) the initial data, and whether a solution is realistic depends on whether the initial data are realistic. Often we can guess a solution and then verify that it satisfies the field equations. Whether it is a solution that could correspond to something in the real world then depends on whether it is associated with initial data that can be found in the real world.

In my thesis I have investigated a few solutions of the string theory equations. These are solutions which are particularly interesting not because they are closely related to the real world, but because they are *exact* solutions, meaning they are not just approximations valid at low energies (which is more often the type of solutions people study).

I have studied a few aspects of these solutions. One is the so-called stringy halo of extended membrane-like objects (D-branes) in an exact solution called the plane wave. Strings can attach to these extended objects, which therefore appear to have string "hair" reaching out – this is the stringy halo.

Another aspect I have studied is the existence of time loops (closed timelike curves) in another solution. The existence of such loops seems to allow time travel, which would then lead to paradoxes of the sort that you could travel back in time and kill your own grandfather. Specifically, I have computed the exact solution and studied how it modifies the picture as compared to the low-energy approximation. The modification turns out to be mild, and the exact solution contains time loops just as the low-energy approximation does. This is a sign that string theory might be perfectly happy with such phenomena, at least to some extent.

B.1 Norwegian translation: Samandrag av avhandlinga for lekmenn

Fysikk

Det ultimate målet med naturvitskap er å forstå alle sider ved naturen. Å forstå menneskenaturen er hovudinteressa i psykologi. Men om du ønskjer å forstå særskilte sider ved menneska på eit djupare, meir grunnleggjande nivå, kjem du til disiplinen kalla biologi. Hovudinteressa her er å forstå liv. Om du enno ikkje er nøgd, men ønskjer å forstå dei underliggjande prinsippa for liv, må du studere ymse kjemiske prosessar og eigenskapar til molekyl og atom. Dette leier deg til domenet åt kjemi. Dersom du held fram med spørsmål om kva atom er, kva dei er sett saman av og korleis dei oppfører seg, kjem du til feltet kalla fysikk, som har som interesse å forstå dei grunnleggjande lovane for partiklar og vekselvirkingane mellom dei. Formuleringa av desse lovane krev matematikk, som er "morsspråket" til fysikk. Dei fysiske lovane blir skrivi som matematiske likningar, og studiet av desse likningane er hovudinteressa for *teoretisk fysikk.*

Ideelt sett kan undersøkingane av desse likningane gjerast i dialog med eksperiment – teoretikaren gjev eksperimentalisten prediksjonar, og eksperimentalisten gjev teoretikaren resultat. Og dei bør helst passe i hop! Det er derimot omstende der eksperimentelle resultat ganske enkelt er utanfor rekkevidde. Da står teoretikaren aleine, og berre intuisjon og matematisk konsistens kan hjelpe han eller henne framover. På sett og vis blir dette mindre fysikk og meir matematikk, og termen *matematisk fysikk* svarar betre til kva denne typen vitskap er.

Forrige århundre var vitne til to banebrytande framsteg i korleis vi forstår naturen på eit fundamentalt fysisk nivå. Den eine var teorien om generell relativitet, som er ein teori for gravitasjon og universet på stor skala. Den andre var kvantefeltteorien, eller Standardmodellen, som er ein teori for elementærpartiklane. Desse to teoriane har vorti testa og stadfesta med imponerande presisjon, og det er liten tvil om at dei korrekt beskriv naturen omkring oss.

Men det er eit problem: Den eine motseier den andre! Vanlegvis er ikkje dette noko problem, sidan usemja er relevant berre i særs eksotiske omstende (*ekstremt* høg energi-tettleik) som sjeldan opptrer i naturen. Men det er ei fundamental usemje som *er* viktig for eksempel for å forstå svarte hol, og Big Bang – byrjinga på universet. Det er ikkje eksperiment eller erfaring som fortel oss at noko manglar i teoriane våre, men teoriane sjølv seier dette.

Søket etter ein teori som klarar opp i denne mangelen på konsistens – ein kvantegravitasjonsteori – har vori det viktigaste langtidsmålet i teoretisk/matematisk fysikk dei siste tiåra. Det er her strengteori entrar scenen.

Strengteori

Den grunnleggjande hypotesa i strengteori er at dei fundamentale byggeklossane ikkje er punkt-partiklar (slik som kvarkar og elektron), men små vibrerande strengar med ei lengd om lag 10^{-34} m. (For å ha ein ide om kva skala dette er, tenk deg ein streng forstørra opp til 1 mm. Eit atom vil da blåsast opp og bli så stor som heile galaksen vår. Alternativt, dersom ein streng var like stor som eit atom, ville eit atom vera like stort som solsystemet.) Ulike vibrasjonar åt strengen svarar til ulike partiklar, liksom ulike vibrasjonar til ein gitarstreng gjev ulike tonar. Dette er den grunnleggjande ideen, men eg må framheve med ein gong at det er langt frå klart akkurat korleis vi kan identifisere dei partiklane vi observerer i naturen (elektron, kvarkar osb.) innan strengteori. Likevel, det som er veldig attraktivt med strengteori er at teorien på ein naturleg måte omfattar både gravitasjon og elementærpartiklar, og er ein kvantegravitasjonsteori som er fri for motsetningar. Hurra! Det einaste problemet er: Korleis får vi den typen gravitasjon og den typen elementærpartiklar vi ser i naturen?

Strengteori har ein ekstremt rik struktur, og på mange måtar kan det vera betre å tenkje på han som eit rammeverk for teoretisk fysikk snarare enn som ein bestemt modell. Ikkje desto mindre, håpet er at strengteori ein dag vil vera så bra forstått at det blir mogleg å forklare opphavet til både gravitasjon og Standardmodellen. For tida blir derimot det meste innan strengteori gjort først og framst med tanke på forstå sjølve teorien (og let vera for framtidige studiar spørsmåla om korleis han svarar til den verkelege verda.)

Sjølv om det ikkje har ført til direkte kontakt med den observerbare verda, har studiet av strengteori gjort at folk har oppdaga interessante relasjonar innan teoretisk fysikk, og til og med matematikk. Slik forstår vi no betre mange fenomen også innan andre (meir tradisjonelle) område av teoretisk fyskk. For eksempel forklarer strengteori at spesielle teoriar for elementærpartiklar (kvantefeltteoriar) og spesielle gravitasjonsteoriar er to sider av same sak. Dette har opna opp nye vindaug for å studere båe sider, og er eit verdifullt resultat uavhengig av korleis eller om i det heile tatt strengteori har noko med den verkelege verda å gjera i seg sjølv.

Avhandlinga

Strengteori, liksom andre modellar i teoretisk fysikk, kjem samans med fundamentale feltlikningar. Desse er differensiallikningar som generelt har mange løysingar, men er vanlegvis vanskelege å løyse. Å løyse differentsiallikningar krev også at vi kjenner initialdata, og kor vidt ei løysing er realistisk er avhengig av om initialdata er realistisk. Ofte kan vi gjette ei løysing og så demonstrere at ho tilfredsstiller fel-likningane. Kor vidt det er ei løysing som har noko med den verkelege verda å gjera er da avhengig av om ho svarar til initialdata som kan finnast i den verkelege verda.

I avhandlinga mi har eg studert nokre løysingar av strengteori-likningane. Desse er løysingar som er spesielt interessante ikkje fordi dei er realistiske, men fordi dei er *eksakte* løysingar. Med det meiner eg at dei ikkje berre er tilnærmingar som er gyldige ved låg energi (som er den typen løysingar som folk oftast studerer).

Eg har studert nokre aspekt ved desse løysingane. Eitt er den såkalla streng-haloen åt membran-liknande objekt (D-branar) i ei eksakt løysing kalla planbølgje-tidrommet. Strengar kan feste seg til slike objekt, som derfor ser ut til å ha streng-"hår" stikkande ut – dette er streng-haloen.

Eit anna aspekt eg har studert er tidssirklar (lukka tidlike kurvar) i ei anna eksakt løysing. Nærveret av slike sirklar synest å tillate tidsreiser, som leier til problematiske paradoks av typen at du kan reise tilbake i tid og skyte din eigen farfar. Spesifikt har eg rekna ut den eksakte løysinga og studert korleis ho modifiserer biletet samanlikna med låg-energi-tilnærminga. Modifikasjonen viser seg å vera mild, og den eksakte løysinga har tidssirklar akkurat som låg-energi-tilnærminga har. Dette er eit teikn på at strengteori kanskje er, i alle fall i nokon grad, lykkeleg og tilfreds trass slike fenomen.

Bibliography

[1] C. V. Johnson and H. G. Svendsen, *A note on D-brane - anti-D-brane interactions in plane wave backgrounds*, *JHEP* **05** (2003) 055 [hep-th/0303255].

[2] C. V. Johnson and H. G. Svendsen, *An exact string theory model of closed time-like curves and cosmological singularities*, hep-th/0405141. Submitted to *Phys. Rev. D.*

[3] G. Veneziano, *Construction of a crossing - symmetric, regge behaved amplitude for linearly rising trajectories*, *Nuovo. Cim.* **A57** (1968) 190–197.

[4] Y. Nambu, *Quark model and the factorization of the veneziano amplitude*, in *Broken symmetry: selected papers of Y. Nambu* (K. Nishijima and T. Eguchi, eds.), no. 13 in 20th Cent. Phys., pp. 258–267. Singapore: World Sci., 1995.

[5] H. B. Nielsen, *An almost physicsl interpretation of the intergrand of the n-point veneziano model*, in *Proceedings of the 15th International Conference on High Energy Physics (Kiev)*, 1970.

[6] L. Susskind, *Dual symmetric theory of hadrons. 1*, *Nuovo Cim.* **A69** (1970) 457–496.

[7] J. Scherk and J. H. Schwarz, *Dual models for nonhadrons*, *Nucl. Phys.* **B81** (1974) 118–144.

[8] M. B. Green and J. H. Schwarz, *Anomaly cancellation in supersymmetric D=10 gauge theory and superstring theory*, *Phys. Lett.* **B149** (1984) 117–122.

[9] J. Polchinski, *Dirichlet-branes and ramond-ramond charges*, *Phys. Rev. Lett.* **75** (1995) 4724–4727 [hep-th/9510017].

[10] B. R. Greene, *The elegant universe: Superstrings, hidden dimensions, and the quest of the ultimate theory.* New York, USA: Norton, 1999. 448p.

[11] M. B. Green, J. H. Schwarz and E. Witten, *Superstring theory. Vol. 1: Introduction.* Cambridge, UK: Univ. Pr., 1987. 469 p.

[12] M. B. Green, J. H. Schwarz and E. Witten, *Superstring theory. Vol. 2: Loop amplitudes, anomalies and phenomenology.* Cambridge, UK: Univ. Pr., 1987. 596 p.

[13] J. Polchinski, *String theory. Vol. 1: An introduction to the bosonic string.* Cambridge, UK: Univ. Pr., 1998. 402 p.

[14] J. Polchinski, *String theory. Vol. 2: Superstring theory and beyond.* Cambridge, UK: Univ. Pr., 1998. 531 p.

[15] C. V. Johnson, *D-Branes.* Cambridge, UK: Univ. Pr., 2003. 548 p.

[16] E. Kiritsis, *Introduction to superstring theory*, hep-th/9709062.

[17] C. V. Johnson, *D-brane primer*, hep-th/0007170.

[18] R. J. Szabo, *Busstepp lectures on string theory: An introduction to string theory and d-brane dynamics*, hep-th/0207142.

[19] T. Mohaupt, *Introduction to string theory*, hep-th/0207249.

[20] J. G. Russo, *Cosmological string models from milne spaces and SL(2,Z) orbifold*, *Mod. Phys. Lett.* **A19** (2004) 421–432 [hep-th/0305032].

[21] A. A. Tseytlin, *Exact solutions of closed string theory*, *Class. Quant. Grav.* **12** (1995) 2365–2410 [hep-th/9505052].

[22] M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, *A new maximally supersymmetric background of iib superstring theory*, *JHEP* **01** (2002) 047 [hep-th/0110242].

[23] R. R. Metsaev, *Type IIB green-schwarz superstring in plane wave Ramond-Ramond background*, *Nucl. Phys.* **B625** (2002) 70–96 [hep-th/0112044].

[24] M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, *Penrose limits and maximal supersymmetry*, *Class. Quant. Grav.* **19** (2002) L87–L95 [hep-th/0201081].

[25] R. Penrose, *Any space-time has a plane wave limit*, in *Differential geometry and relativity* (M. Cahen and M. Flato, eds.), pp. 271 – 275. Dordrecht, Holland: Reidel Pub., 1976.

[26] R. Güven, *Plane wave limits and T-duality*, *Phys. Lett.* **B482** (2000) 255–263 [hep-th/0005061].

[27] D. Brecher, C. V. Johnson, K. J. Lovis and R. C. Myers, *Penrose limits, deformed pp-waves and the string duals of $N = 1$ large N gauge theory*, *JHEP* **10** (2002) 008 [hep-th/0206045].

[28] D. Berenstein, J. M. Maldacena and H. Nastase, *Strings in flat space and pp waves from $N = 4$ super yang mills*, *JHEP* **04** (2002) 013 [hep-th/0202021].

[29] J. M. Maldacena, *The large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231–252 [hep-th/9711200].

[30] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B428** (1998) 105–114 [hep-th/9802109].

[31] E. Witten, *Anti-de sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253–291 [hep-th/9802150].

[32] G. 't Hooft, *A planar diagram theory for strong interactions*, *Nucl. Phys.* **B72** (1974) 461.

[33] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, *Large N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183–386 [hep-th/9905111].

[34] J. L. Petersen, *Introduction to the maldacena conjecture on AdS/CFT*, *Int. J. Mod. Phys.* **A14** (1999) 3597–3672 [hep-th/9902131].

[35] P. Di Vecchia, *An introduction to AdS/CFT correspondence*, *Fortsch. Phys.* **48** (2000) 87–92 [hep-th/9903007].

[36] J. C. Plefka, *Lectures on the plane-wave string / gauge theory duality*, *Fortsch. Phys.* **52** (2004) 264–301 [hep-th/0307101].

[37] K. Skenderis and M. Taylor, *Branes in AdS and pp-wave spacetimes*, *JHEP* **06** (2002) 025 [hep-th/0204054].

[38] M. R. Gaberdiel and M. B. Green, *The D-instanton and other supersymmetric D-branes in IIB plane-wave string theory*, *Ann. Phys.* **307** (2003) 147–194 [hep-th/0211122].

[39] M. Billo and I. Pesando, *Boundary states for GS superstrings in an Hpp wave background*, *Phys. Lett.* **B536** (2002) 121–128 [hep-th/0203028].

[40] A. Dabholkar and S. Parvizi, *Dp branes in pp-wave background*, *Nucl. Phys.* **B641** (2002) 223–234 [hep-th/0203231].

[41] O. Bergman, M. R. Gaberdiel and M. B. Green, *D-brane interactions in type IIB plane-wave background*, *JHEP* **03** (2003) 002 [hep-th/0205183].

[42] M. R. Gaberdiel and M. B. Green, *D-branes in a plane-wave background*, *Fortsch. Phys.* **51** (2003) 713–718 [hep-th/0212052].

[43] K. Skenderis and M. Taylor, *Open strings in the plane wave background. I: quantization and symmetries*, *Nucl. Phys.* **B665** (2003) 3–48 [hep-th/0211011].

[44] K. Skenderis and M. Taylor, *Open strings in the plane wave background. ii: Superalgebras and spectra*, *JHEP* **07** (2003) 006 [hep-th/0212184].

[45] K. Skenderis and M. Taylor, *An overview of branes in the plane wave background*, *Class. Quant. Grav.* **20** (2003) S567–S574 [hep-th/0301221].

[46] J. J. Atick and E. Witten, *The hagedorn transition and the number of degrees of freedom of string theory*, *Nucl. Phys.* **B310** (1988) 291–334.

[47] B. R. Greene, K. Schalm and G. Shiu, *On the hagedorn behaviour of pp-wave strings and $N = 4$ sym theory at finite R-charge density*, *Nucl. Phys.* **B652** (2003) 105–126 [hep-th/0208163].

[48] T. Banks and L. Susskind, *Brane - antibrane forces*, hep-th/9511194.

[49] A. Sen, *Stable non-BPS states in string theory*, *JHEP* **06** (1998) 007 [hep-th/9803194].

[50] A. Sen, *Stable non-BPS bound states of BPS D-branes*, *JHEP* **08** (1998) 010 [hep-th/9805019].

[51] A. Sen, *Tachyon condensation on the brane antibrane system*, *JHEP* **08** (1998) 012 [hep-th/9805170].

[52] A. Sen, *SO(32) spinors of type I and other solitons on brane- antibrane pair*, *JHEP* **09** (1998) 023 [hep-th/9808141].

[53] R. Minasian and G. W. Moore, *K-theory and ramond-ramond charge*, *JHEP* **11** (1997) 002 [hep-th/9710230].

[54] E. Witten, *D-branes and K-theory*, *JHEP* **12** (1998) 019 [hep-th/9810188].

[55] P. Horava, *Type iia d-branes, k-theory, and matrix theory*, *Adv. Theor. Math. Phys.* **2** (1999) 1373–1404 [hep-th/9812135].

[56] D. Amati and C. Klimcik, *Nonperturbative computation of the weyl anomaly for a class of nontrivial backgrounds*, *Phys. Lett.* **B219** (1989) 443.

[57] G. T. Horowitz and A. R. Steif, *Space-time singularities in string theory*, *Phys. Rev. Lett.* **64** (1990) 260.

[58] G. T. Horowitz and A. R. Steif, *Strings in strong gravitational fields*, *Phys. Rev.* **D42** (1990) 1950–1959.

[59] O. Jofre and C. Nunez, *Strings in plane wave backgrounds revisited*, *Phys. Rev.* **D50** (1994) 5232–5240 [hep-th/9311187].

[60] M. B. Green and M. Gutperle, *Effects of D-instantons*, *Nucl. Phys.* **B498** (1997) 195–227 [hep-th/9701093].

[61] A. Dabholkar and J. Raeymaekers, *Comments on D-brane interactions in pp-wave backgrounds*, *JHEP* **11** (2003) 032 [hep-th/0309039].

[62] C. A. R. Herdeiro, *Special properties of five dimensional BPS rotating black holes*, *Nucl. Phys.* **B582** (2000) 363–392 [hep-th/0003063].

[63] M. Cvetic and D. Youm, *General rotating five dimensional black holes of toroidally compactified heterotic string*, *Nucl. Phys.* **B476** (1996) 118–132 [hep-th/9603100].

[64] J. C. Breckenridge, R. C. Myers, A. W. Peet and C. Vafa, *D-branes and spinning black holes*, *Phys. Lett.* **B391** (1997) 93–98 [hep-th/9602065].

[65] L. Järv and C. V. Johnson, *Rotating black holes, closed time-like curves, thermodynamics, and the enhancon mechanism*, *Phys. Rev.* **D67** (2003) 066003 [hep-th/0211097].

[66] T. H. Buscher, *A symmetry of the string background field equations*, *Phys. Lett.* **B194** (1987) 59.

[67] T. H. Buscher, *Path integral derivation of quantum duality in nonlinear sigma models*, *Phys. Lett.* **B201** (1988) 466.

[68] P. Meessen and T. Ortin, *An sl(2,z) multiplet of nine-dimensional type ii supergravity theories*, *Nucl. Phys.* **B541** (1999) 195–245 [hep-th/9806120].

[69] C. A. R. Herdeiro, *Spinning deformations of the D1-D5 system and a geometric resolution of closed timelike curves*, *Nucl. Phys.* **B665** (2003) 189–210 [hep-th/0212002].

[70] C. V. Johnson, A. W. Peet and J. Polchinski, *Gauge theory and the excision of repulson singularities*, *Phys. Rev.* **D61** (2000) 086001 [hep-th/9911161].

[71] S. Mukhi, M. Rangamani and E. Verlinde, *Strings from quivers, membranes from moose*, *JHEP* **05** (2002) 023 [hep-th/0204147].

[72] J. L. F. Barbón and E. Rabinovici, *Stringy fuzziness as the custodian of time-space noncommutativity*, *Phys. Lett.* **B486** (2000) 202–211 [hep-th/0005073].

[73] K. S. Thorne, *Closed timelike curves*, in *Proceedings of the 13th international conference on general relativity and gravitation, Cordoba, Argentina 1992* (R. J. Gleiser, C. N. Kozameh and O. Moreschi, eds.), pp. 295 – 315, Bristol, UK: IoP Publishing, 1993.

[74] S. W. Hawking, *The chronology protection conjecture*, *Phys. Rev.* **D46** (1992) 603–611.

[75] M. Visser, *The quantum physics of chronology protection*, gr-qc/0204022.

[76] G. W. Gibbons and C. A. R. Herdeiro, *Supersymmetric rotating black holes and causality violation*, *Class. Quant. Grav.* **16** (1999) 3619–3652 [hep-th/9906098].

[77] L. Dyson, *Chronology protection in string theory*, *JHEP* **03** (2004) 024 [hep-th/0302052].

[78] E. G. Gimon and P. Horava, *Over-rotating black holes, goedel holography and the hypertube*, hep-th/0405019.

[79] E. K. Boyda, S. Ganguli, P. Horava and U. Varadarajan, *Holographic protection of chronology in universes of the goedel type*, *Phys. Rev.* **D67** (2003) 106003 [hep-th/0212087].

[80] D. Brecher, P. A. DeBoer, D. C. Page and M. Rozali, *Closed timelike curves and holography in compact plane waves*, *JHEP* **10** (2003) 031 [hep-th/0306190].

[81] V. E. Hubeny, M. Rangamani and S. F. Ross, *Causal inheritance in plane wave quotients*, *Phys. Rev.* **D69** (2004) 024007 [hep-th/0307257].

[82] D. Brace, *Closed geodesics on goedel-type backgrounds*, *JHEP* **12** (2003) 021 [hep-th/0308098].

[83] C. M. Hull, *Timelike T-duality, de sitter space, large N gauge theories and topological field theory*, *JHEP* **07** (1998) 021 [hep-th/9806146].

[84] C. M. Hull, *Duality and the signature of space-time*, *JHEP* **11** (1998) 017 [hep-th/9807127].

[85] E. Witten, *Nonabelian bosonization in two dimensions*, *Commun. Math. Phys.* **92** (1984) 455–472.

[86] A. M. Polyakov and P. B. Wiegmann, *Theory of nonabelian goldstone bosons in two dimensions*, *Phys. Lett.* **B131** (1983) 121–126.

[87] S. P. Novikov, *The hamiltonian formalism and a many valued analog of morse theory*, *Usp. Mat. Nauk* **37N5** (1982) 3–49.

[88] P. Di Francesco, P. Mathieu and D. Senechal, *Conformal field theory.* New York, USA: Springer, 1997. 890 p.

[89] P. H. Ginsparg, *Applied conformal field theory*, hep-th/9108028. Published in Les Houches Summer School 1988:1-168.

[90] K. Bardakci and M. B. Halpern, *New dual quark models*, *Phys. Rev.* **D3** (1971) 2493.

[91] M. B. Halpern, *The two faces of a dual pion - quark model*, *Phys. Rev.* **D4** (1971) 2398.

[92] P. Goddard, A. Kent and D. I. Olive, *Virasoro algebras and coset space models*, *Phys. Lett.* **B152** (1985) 88.

[93] P. Goddard, A. Kent and D. I. Olive, *Unitary representations of the virasoro and supervirasoro algebras*, *Commun. Math. Phys.* **103** (1986) 105.

[94] E. Witten, *On holomorphic factorization of WZW and coset models*, *Commun. Math. Phys.* **144** (1992) 189–212.

[95] K. Bardakci, E. Rabinovici and B. Saering, *String models with c ¡ 1 components*, *Nucl. Phys.* **B299** (1988) 151.

[96] K. Gawedzki and A. Kupiainen, *G/H conformal field theory from gauged WZW model*, *Phys. Lett.* **B215** (1988) 119–123.

[97] K. Gawedzki and A. Kupiainen, *Coset construction from functional integrals*, *Nucl. Phys.* **B320** (1989) 625.

[98] D. Karabali, Q.-H. Park, H. J. Schnitzer and Z. Yang, *A GKO construction based on a path integral formulation of gauged wess-zumino-witten actions*, *Phys. Lett.* **B216** (1989) 307.

[99] P. Bowcock, *Canonical quantization of the gauged wess-zumino model*, *Nucl. Phys.* **B316** (1989) 80.

[100] D. Karabali and H. J. Schnitzer, *BRST quantization of the gauged WZW action and coset conformal field theories*, *Nucl. Phys.* **B329** (1990) 649.

[101] V. G. Knizhnik and A. B. Zamolodchikov, *Current algebra and wess-zumino model in two dimensions*, *Nucl. Phys.* **B247** (1984) 83–103.

[102] D. Gepner and E. Witten, *String theory on group manifolds*, *Nucl. Phys.* **B278** (1986) 493.

[103] P. Goddard and D. I. Olive, *Kac-moody and virasoro algebras in relation to quantum physics*, *Int. J. Mod. Phys.* **A1** (1986) 303.

[104] E. Witten, *On string theory and black holes*, *Phys. Rev.* **D44** (1991) 314–324.

[105] R. Dijkgraaf, H. Verlinde and E. Verlinde, *String propagation in a black hole geometry*, *Nucl. Phys.* **B371** (1992) 269–314.

[106] A. A. Tseytlin, *On the form of the black hole solution in $d = 2$ theory*, *Phys. Lett.* **B268** (1991) 175–178.

[107] I. Jack, D. R. T. Jones and J. Panvel, *Exact bosonic and supersymmetric string black hole solutions*, *Nucl. Phys.* **B393** (1993) 95–110 [hep-th/9201039].

[108] I. Bars and K. Sfetsos, *Conformally exact metric and dilaton in string theory on curved space-time*, *Phys. Rev.* **D46** (1992) 4510–4519 [hep-th/9206006].

[109] I. Bars and K. Sfetsos, *$sl(2,\mathrm{R}) \times su(2)/\mathrm{R}^2$ string model in curved space-time and exact conformal results*, *Phys. Lett.* **B301** (1993) 183–190 [hep-th/9208001].

[110] A. A. Tseytlin, *Effective action of gauged wzw model and exact string solutions*, *Nucl. Phys.* **B399** (1993) 601–622 [hep-th/9301015].

[111] I. Bars and K. Sfetsos, *Exact effective action and space-time geometry in gauged WZW models*, *Phys. Rev.* **D48** (1993) 844–852 [hep-th/9301047].

[112] H. Leutwyler and M. A. Shifman, *Perturbation theory in the wess-zumino-novikov-witten model*, *Int. J. Mod. Phys.* **A7** (1992) 795–842.

[113] M. A. Shifman, *Four-dimension aspect of the perturbative renormalization in three-dimensional chern-simons theory*, *Nucl. Phys.* **B352** (1991) 87–112.

[114] A. M. Polyakov and P. B. Wiegmann, *Goldstone fields in two-dimensions with multivalued actions*, *Phys. Lett.* **B141** (1984) 223–228.

[115] C. V. Johnson, *Exact models of extremal dyonic 4-D black hole solutions of heterotic string theory*, *Phys. Rev.* **D50** (1994) 4032–4050 [hep-th/9403192].

[116] T. Banks, L. J. Dixon, D. Friedan and E. J. Martinec, *Phenomenology and conformal field theory or can string theory predict the weak mixing angle?*, *Nucl. Phys.* **B299** (1988) 613–626.

[117] A. Sen, *(2, 0) supersymmetry and space-time supersymmetry in the heterotic string theory*, *Nucl. Phys.* **B278** (1986) 289.

[118] E. Witten, *The N matrix model and gauged WZW models*, *Nucl. Phys.* **B371** (1992) 191–245.

[119] K. Sfetsos and A. A. Tseytlin, *Four-dimensional plane wave string solutions with coset CFT description*, *Nucl. Phys.* **B427** (1994) 245–272 [hep-th/9404063].

[120] J. Maharana and J. H. Schwarz, *Noncompact symmetries in string theory*, *Nucl. Phys.* **B390** (1993) 3–32 [hep-th/9207016].

[121] A. H. Taub, *Empty space-times admitting a three parameter group of motions*, *Annals Math.* **53** (1951) 472–490.

[122] E. Newman, L. Tamubrino and T. Unti, *Empty space generalization of the schwarzschild metric*, *J. Math. Phys.* **4** (1963) 915.

[123] S. W. Hawking and G. F. R. Ellis, *The large scale structure of space-time.* Cambridge, UK: Univ. Pr., 1973. 391 p.

[124] C. W. Misner in *Relativity Theory and Astrophysics I: Relativity and Cosmology* (J. Ehlers, ed.), vol. 8 of *Lectures in Applied Mathematics.* American Mathematical Society, Providence, 1967.

[125] W. A. Hiscock and D. A. Konkowski, *Quantum vacuum energy in taub - nut (newman-unti-tamburino) type cosmologies*, *Phys. Rev.* **D26** (1982) 1225–1230.

[126] S. W. Kim and K. P. Thorne, *Do vacuum fluctuations prevent the creation of closed timelike curves?*, *Phys. Rev.* **D43** (1991) 3929–3947.

[127] L.-X. Li, J.-M. Xu and L. Liu, *Complex geometry, quantum tunneling, and time machines*, *Phys. Rev.* **D48** (1993) 4735–4737.

[128] L.-X. Li, *New light on time machines: Against the chronology protection conjecture*, *Phys. Rev.* **D50** (1994) 6037–6040.

[129] T. Tanaka and W. A. Hiscock, *Massive scalar field in multiply connected flat space- times*, *Phys. Rev.* **D52** (1995) 4503–4511 [gr-qc/9504021].

[130] L.-X. Li, *Must time machines be unstable against vacuum fluctuations?*, *Class. Quant. Grav.* **13** (1996) 2563–2568 [gr-qc/9703024].

[131] S. V. Krasnikov, *On the quantum stability of the time machine*, *Phys. Rev.* **D54** (1996) 7322–7327 [gr-qc/9508038].

[132] S. V. Sushkov, *Chronology protection and quantized fields: Complex automorphic scalar field in misner space*, *Class. Quant. Grav.* **14** (1997) 523–534 [gr-qc/9509056].

[133] M. Visser, *Traversable wormholes: The roman ring*, *Phys. Rev.* **D55** (1997) 5212–5214 [gr-qc/9702043].

[134] L.-X. Li and I. Gott, J. Richard, *A self-consistent vacuum for misner space and the chronology protection conjecture*, *Phys. Rev. Lett.* **80** (1998) 2980 [gr-qc/9711074].

[135] R. Biswas, E. Keski-Vakkuri, R. G. Leigh, S. Nowling and E. Sharpe, *The taming of closed time-like curves*, *JHEP* **01** (2004) 064 [hep-th/0304241].

[136] J. Khoury, B. A. Ovrut, N. Seiberg, P. J. Steinhardt and N. Turok, *From big crunch to big bang*, *Phys. Rev.* **D65** (2002) 086007 [hep-th/0108187].

[137] H. Liu, G. Moore and N. Seiberg, *Strings in a time-dependent orbifold*, *JHEP* **06** (2002) 045 [hep-th/0204168].

[138] H. Liu, G. Moore and N. Seiberg, *Strings in time-dependent orbifolds*, *JHEP* **10** (2002) 031 [hep-th/0206182].

[139] J. Simon, *The geometry of null rotation identifications*, *JHEP* **06** (2002) 001 [hep-th/0203201].

[140] A. Lawrence, *On the instability of 3D null singularities*, *JHEP* **11** (2002) 019 [hep-th/0205288].

[141] G. T. Horowitz and J. Polchinski, *Instability of spacelike and null orbifold singularities*, *Phys. Rev.* **D66** (2002) 103512 [hep-th/0206228].

[142] P. Horava, *Some exact solutions of string theory in four-dimensions and five-dimensions*, *Phys. Lett.* **B278** (1992) 101–110 [hep-th/9110067].

[143] C. R. Nappi and E. Witten, *A closed, expanding universe in string theory*, *Phys. Lett.* **B293** (1992) 309–314 [hep-th/9206078].

[144] A. Giveon and A. Pasquinucci, *On cosmological string backgrounds with toroidal isometries*, *Phys. Lett.* **B294** (1992) 162–166 [hep-th/9208076].

[145] E. Kiritsis and C. Kounnas, *Dynamical topology change in string theory*, *Phys. Lett.* **B331** (1994) 51–62 [hep-th/9404092].

[146] C. Kounnas and D. Lust, *Cosmological string backgrounds from gauged WZW models*, *Phys. Lett.* **B289** (1992) 56–60 [hep-th/9205046].

[147] A. A. Tseytlin and C. Vafa, *Elements of string cosmology*, *Nucl. Phys.* **B372** (1992) 443–466 [hep-th/9109048].

[148] H. J. de Vega, A. L. Larsen and N. Sanchez, *Nonsingular string cosmologies from exact conformal field theories*, *Nucl. Phys. Proc. Suppl.* **102** (2001) 201–208 [hep-th/0110262].

[149] S. Elitzur, A. Giveon, D. Kutasov and E. Rabinovici, *From big bang to big crunch and beyond*, *JHEP* **06** (2002) 017 [hep-th/0204189].

[150] L. Cornalba and M. S. Costa, *A new cosmological scenario in string theory*, *Phys. Rev.* **D66** (2002) 066001 [hep-th/0203031].

[151] B. Craps, D. Kutasov and G. Rajesh, *String propagation in the presence of cosmological singularities*, *JHEP* **06** (2002) 053 [hep-th/0205101].

[152] A. Buchel, P. Langfelder and J. Walcher, *On time-dependent backgrounds in supergravity and string theory*, *Phys. Rev.* **D67** (2003) 024011 [hep-th/0207214].

[153] M. Fabinger and J. McGreevy, *On smooth time-dependent orbifolds and null singularities*, *JHEP* **06** (2003) 042 [hep-th/0206196].

[154] L. Cornalba, M. S. Costa and C. Kounnas, *A resolution of the cosmological singularity with orientifolds*, *Nucl. Phys.* **B637** (2002) 378–394 [hep-th/0204261].

[155] L. Cornalba and M. S. Costa, *On the classical stability of orientifold cosmologies*, *Class. Quant. Grav.* **20** (2003) 3969–3996 [hep-th/0302137].

[156] M. Berkooz, B. Craps, D. Kutasov and G. Rajesh, *Comments on cosmological singularities in string theory*, *JHEP* **03** (2003) 031 [hep-th/0212215].

[157] M. Berkooz, B. Pioline and M. Rozali, *Closed strings in misner space: Cosmological production of winding strings*, *JCAP* **0408** (2004) 004 [hep-th/0405126].

[158] B. Fiol, C. Hofman and E. Lozano-Tellechea, *Causal structure of $d = 5$ vacua and axisymmetric spacetimes*, *JHEP* **02** (2004) 034 [hep-th/0312209].

[159] L. Maoz and J. Simon, *Killing spectroscopy of closed timelike curves*, *JHEP* **01** (2004) 051 [hep-th/0310255].

[160] D. Brace, *Over-rotating supertube backgrounds*, hep-th/0310186.

[161] D. Brace, C. A. R. Herdeiro and S. Hirano, *Classical and quantum strings in compactified pp-waves and goedel type universes*, *Phys. Rev.* **D69** (2004) 066010 [hep-th/0307265].

[162] K. Behrndt and M. Possel, *Chronological structure of a goedel type universe with negative cosmological constant*, *Phys. Lett.* **B580** (2004) 1–6 [hep-th/0310090].

[163] N. Drukker, B. Fiol and J. Simon, *Goedel-type universes and the landau problem*, hep-th/0309199.

[164] D. Brecher, U. H. Danielsson, J. P. Gregory and M. E. Olsson, *Rotating black holes in a goedel universe*, *JHEP* **11** (2003) 033 [hep-th/0309058].

[165] Y. Hikida and S.-J. Rey, *Can branes travel beyond ctc horizon in goedel universe?*, *Nucl. Phys.* **B669** (2003) 57–77 [hep-th/0306148].

[166] I. Bena and P. Kraus, *Three charge supertubes and black hole hair*, hep-th/0402144.

[167] K. Gödel, *An example of a new type of cosmological solutions of einstein's field equations of graviation*, *Rev. Mod. Phys.* **21** (1949) 447–450.

[168] E. G. Gimon and A. Hashimoto, *Black holes in goedel universes and pp-waves*, *Phys. Rev. Lett.* **91** (2003) 021601 [hep-th/0304181].

[169] N. Drukker, B. Fiol and J. Simon, *Goedel's universe in a supertube shroud*, *Phys. Rev. Lett.* **91** (2003) 231601 [hep-th/0306057].

[170] D. Israel, *Quantization of heterotic strings in a goedel/anti de sitter spacetime and chronology protection*, *JHEP* **01** (2004) 042 [hep-th/0310158].

[171] N. Drukker, *Supertube domain-walls and elimination of closed time-like curves in string theory*, hep-th/0404239.

[172] C. V. Johnson and R. C. Myers, *Taub-NUT dyons in heterotic string theory*, *Phys. Rev.* **D50** (1994) 6512–6518 [hep-th/9406069].

[173] R. Kallosh, D. Kastor, T. Ortin and T. Torma, *Supersymmetry and stationary solutions in dilaton axion gravity*, *Phys. Rev.* **D50** (1994) 6374–6384 [hep-th/9406059].

[174] C. V. Johnson and R. C. Myers, *Stringy twists of the Taub - NUT metric*, hep-th/9409177.

[175] C. V. Johnson, *Heterotic coset models*, *Mod. Phys. Lett.* **A10** (1995) 549–560 [hep-th/9409062].

[176] C. V. Johnson, *Heterotic cosets*, hep-th/9409061.

[177] P. Berglund, C. V. Johnson, S. Kachru and P. Zaugg, *Heterotic coset models and (0,2) string vacua*, *Nucl. Phys.* **B460** (1996) 252–298 [hep-th/9509170].

[178] S. B. Giddings, J. Polchinski and A. Strominger, *Four-dimensional black holes in string theory*, *Phys. Rev.* **D48** (1993) 5784–5797 [hep-th/9305083].

[179] M. J. Perry and E. Teo, *Nonsingularity of the exact two-dimensional string black hole*, *Phys. Rev. Lett.* **70** (1993) 2669–2672 [hep-th/9302037].

[180] E. B. Kiritsis, *Duality in gauged WZW models*, *Mod. Phys. Lett.* **A6** (1991) 2871–2880.

[181] H. Ooguri and C. Vafa, *Two-dimensional black hole and singularities of CY manifolds*, *Nucl. Phys.* **B463** (1996) 55–72 [hep-th/9511164].

[182] D. Tong, *NS5-branes, T-duality and worldsheet instantons*, *JHEP* **07** (2002) 013 [hep-th/0204186].

[183] I. Gott, J. Richard and L.-X. Li, *Can the universe create itself?*, *Phys. Rev.* **D58** (1998) 023501 [astro-ph/9712344].

[184] D. V. Galtsov and O. V. Kechkin, *Ehlers-harrison type transformations in dilaton - axion gravity*, *Phys. Rev.* **D50** (1994) 7394–7399 [hep-th/9407155].

[185] C. V. Johnson and R. C. Myers, *A conformal field theory of a rotating dyon*, *Phys. Rev.* **D52** (1995) 2294–2312 [hep-th/9503027].

[186] R. M. Wald, *General Relativity.* Chicago, USA: Univ. Pr., 1984. 491 p.

[187] T. Takayanagi, *Modular invariance of strings on pp-waves with RR-flux*, *JHEP* **12** (2002) 022 [hep-th/0206010].